Macroeconomic Model and the Missing Equation

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Abstract

We address the missing equation difficulty of macroeconomic models, which is usually associated with the Keynesian model alone, but the "Classical" model is shown to suffer from essentially the same problem. The difficulty of the classical model is resolved by re-examining the behavioral hypothesis under uncertainty by using the Certainty Equivalence of uncertain variables. The Certainty Equivalence concept is also useful to resolve the missing equation difficulty of the Keynesian model, although production function must be simultaneously taken into account. The final section extends the analysis to the uncertainty of the future income.

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1. Introduction: The Missing Equation Problem

The short-run equilibrium (with the capital stock K being fixed at $\bar{K}$) of the conventional macro-economic (IS-LM) model is described as:

$$y = C(y, r) + I(r) \quad \cdots\cdots \quad (1)$$

$$\frac{M}{p} = M^d(y, r) \quad \cdots\cdots \quad (2),$$

where the endogenous variable candidates are y (real GDP), r (the rate of interest) and p (the general price level). With only two equilibrium conditions (1) and (2), it is impossible to determine all the unknowns.

Conventional procedure to deal with the missing equation problem has been to increase the number of equations. The Keynesian model, often postulating the "wage/price rigidity", additionally assumes that

$$p = \bar{p} \quad \cdots\cdots \quad (3\text{-}1),$$

and determines y and r simultaneously.

The "Classical Theory", in contrast, conventionally assumes, in addition to (1) and (2), the following:

$$L^d(\omega) = L^s(\omega) \quad \cdots\cdots \quad (3\text{-}2)$$

$$F_L = \omega \quad \cdots\cdots \quad (3\text{-}3)$$

$$y = F(L^*, \bar{K}) \quad \cdots\cdots \quad (3\text{-}4),$$

where ω is the real wage rate, L^d the demand function for labor, L^s the supply function of labor, L^* the labor force employed at the labor market equilibrium and F the production function. Then the classical theory does not seem to suffer from the missing equation difficulty.

It must be recognized though that the conventional procedure to deal with the missing equation problem has not at all paid attention to the fact that r, the rate of return variable appearing in the money demand function M^d, should be treated as stochastic in the Keynesian framework. It should be so simply because the liquidity preference theory, one of the key elements of the Keynesian model, presupposes that M^d is affected by the uncertain rate of return from assets. The expression $M^d(y, r)$, which writes M^d as directly dependent upon r, should then be logically reexamined carefully, because such an expression automatically renders the money demand itself as stochastic. The conventional IS-LM type analysis in the Keynesian framework, where a fixed LM curve is drawn over the r – y space, is no longer possible once r is regarded as stochastic. It is in fact impossible to draw a fixed IS curve as well, because (1) assumes both investment and consumption as dependent on the stochastic r.

It must be recognized further that the conventional comparison between the Keynes type model composed by (1) and (2), and the Classical model composed by (1), (2), (3-2), (3-3) and (3-4), is inappropriate, as long as both models treat the rate of return variable r as non-stochastic. Although it is true that the Classical model composed as above does not suffer from the missing equation difficulty if the rate of return variable r is treated as non-stochastic, it is inappropriate to compare this model with the Keynesian model simply because the rate of return variable r in the latter must be treated as stochastic: to do otherwise is to compare the Classical model with something quite different from the Keynesian model. To be sure, the comparison should be carried out under uncertainty with respect at least to the financial sector (2), treating the variable r (presumably the rate of return on the risky asset) as stochastic. The appropriate framework to discuss the missing equation issue is to assume uncertainty. The purpose of this paper is to analyze the missing equation problem from this perspective.

As we shall show below, the introduction of uncertainty forces us to modify the specification of behavioral equations, in particular the money demand function. We suggest by this paper that it is no more appropriate to specify the money demand function simply as $M^d(y, r)$[1] under uncertainty. We shall show that the money demand function under uncertainty should be modified as a function of ρ, the certainty equivalence of the rate of return on the overall mixed asset (which itself is optimally composed by risky and non-risky assets).

We will also show that the investment function as well as the consumption function should be appropriately re-specified under uncertainty. Without these re-specifications, and without an additional equation that defines the certainty equivalence, it is impossible

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[1] This specification almost implies that the demand for money stock itself is stochastic when r is stochastic, an implication which does not seem to be well recognized. Although it is clear that the money demand is associated with "r" as a stochastic variable, it is also clear that the optimum money demand under the portfolio selection theory is deterministically chosen. In what follows, we will distinguish between the stochastic variable r on the one hand, and the money demand as a deterministic variable on the other.

to determine all the unknown variables endogenously even under the Classical Theory. In our view, then, the missing equation difficulty is a logical problem not only of the Keynesian model but also of the conventional (or, traditional) classical model itself.

In what follows, Section 2 will consider the modifications of behavioral equations under uncertainty and introduce what we shall call the Certainty Equivalence Method. In Section 3, we apply our approach and consider what we shall call the Modified Classical Model. We will show that this model, unlike the traditional classical model, does not suffer from the missing equation difficulty.

In Section 4, we apply our approach to the Keynesian model and examine if the missing equation difficulty is solved as well. We will show that the Certainty Equivalence Method is necessary but not sufficient to solve the difficulty. We will show, in particular, that if a specific production function relationship is assumed together with the Certainty Equivalence Method, the missing equation difficulty in the Keynesian model will disappear.

Up until Section 5, we restrict our uncertainty to the "Financial Uncertainty", i.e., the uncertainty of the risky financial asset. In Section 5, we introduce the "Future Income Uncertainty" and show that our Certainty Equivalence approach is also useful to this case.

2. Money Demand, Consumption and Investment Functions under Uncertainty

(Money Demand)

In what follows, we assume that the demand for money is derived by the portfolio selection theory[2]. We regard the variable r as the rate of return of the risky asset (which we shall call "equity"), while the rate of return of the safe asset (which we shall call "money") is described as i, an exogenous variable regarded as non-stochastic. We have suggested above that the money demand equation should be modified as a function of ρ, the certainty equivalence of the rate of return on the overall mixed asset (which itself is optimally composed by risky and non-risky assets). We now explain why it should be so.

Consider the equity whose stochastic rate of return "r" is defined in relation to its market price p_s such that:

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[2] For simplicity, we do not introduce inflationary expectations throughout this paper.

$$r \equiv \frac{d+(p_s^e - p_s)}{p_s} \quad \cdots\cdots\cdots\cdots \quad (4),$$

where p_s^e, the expected price of the equity in the next period, is stochastic. It is because p_s^e is stochastic that the variable "r" is stochastic. For simplicity, the value of d (the dividend per share) is assumed as deterministically constant. The term p_s^e is the only source of uncertainty in the monetary sector of our model. The distribution of p_s^e (a subjective distribution as perceived by an agent who faces with the portfolio decision making), and hence its parameters (i.e., each and every moment), are assumed as given in the short run. Because the parameters of the distribution of p_s^e are given, the equation (4) shows that the parameter of the distribution of "r" is a function of p_s. Notice at this stage that introducing uncertainty into the model by assuming the distribution of p_s^e immediately implies that another endogenous variable, p_s, is introduced in addition to the endogenous variables already included in the model (1) and (2).

The agent who makes portfolio decision, knowing the parameters of "r", and using the von-Neumann – Morgenstern (NM) utility function Z[3], will maximize the expected utility EZ of the return from the overall non-human assets (μ), which is defined as:

$$\mu \equiv \alpha r + (1-\alpha)i.$$

The optimum share of the risky asset, α, is chosen by solving the portfolio optimization problem

$$\underset{\alpha}{\mathrm{Max}}\ E[Z(\mu)]^{4}.$$

It must be recognized at this stage that, because there is one-to-one correspondence[5] between the stochastic variable μ and its Certainty Equivalence $Z^{-1}EZ(\mu)$, an agent who maximizes the expected utility is simultaneously maximizing the utility of the Certainty Equivalence of μ.

Suppose this optimization problem has been solved and that α^* is the optimum share of the risky asset. Then $\mu^* = \alpha^* r + (1-\alpha^*)i$ is the rate of return from the optimally mixed non-human assets, and μ^* is again a stochastic variable. However, the share of the risky

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[3] Z(x) is a single-variable NM function that evaluates the NM utility of a stochastic variable x with which the agent faces.

[4] The term μ is the amount of pecuniary returns from a unit of the non-human asset. EZ(μ) is the expected utility of that return.

[5] Notice that Z, being the NM utility function, is monotonic.

asset is now α^*, which is a deterministic value. Hnece, given the non-human wealth stock, the demand for its risky component as well as the demand for money (the non-risky component) is deterministic.

The demand for money, then, is essentially a function in terms of the parameters of the stochastic variable μ^*(and of the income variable y). Therefore, the money demand function $M^d(y, r)$, should more appropriately be written as:

$$M^d(y, \text{parameters of the distribution of } \mu^*),$$

where the parameters of the distribution of μ^* are functions in terms of p_s. Furthermore, considering the one-to-one correspondence between the stochastic variable μ^* and its Certainty Equivalence $Z^{-1}EZ(\mu^*)$, the money demand function above is equivalently written as:

$$M^d(y, \rho)$$

where ρ is defined as $\rho \equiv Z^{-1}EZ(\mu^*)$.

Given the mathematical transform $Z^{-1}EZ$, the value of ρ is determined by the parameters of the distribution of μ^*. Because these parameters are functions in terms of p_s, we may regard ρ itself as a function of p_s, such as

$$\rho = \hat{\rho}(p_s).$$

In conclusion, the money demand function under uncertainty, given the distribution of p_s^e, is to be specified as $M^d(y, \hat{\rho}(p_s))$, not as $M^d(y, r)$, when r refers to the uncertain returns from the risky asset. In order to examine the implication of this re-specification for the missing equation problem, however, we must re-examine further the specification of the flow behavioral equations under uncertainty.

(Consumption and Investment)

Let us next consider how one should re-specify the flow behavioral functions, the consumption demand and the investment demand. The consumption demand, which has been specified above as $C(y, r)$, depends on the income variable y and on the interest rate variable r, both of which could be stochastic. We shall deal in Section 5 with the uncertainty with respect to y, while this section deals with the uncertainty with respect to r.

Let us first note that the rate of return from the over-all saving stock is not r (i.e., the rate of return on the risky asset), but μ^*, the (optimized) rate of return of the overall nonhuman-asset, defined as $\mu^* = \alpha^* r + (1 - \alpha^*)i$. It is then evident that the consumption (or, saving) decision making is dependent on μ^*, not r. The consumption function above, i.e., C(y, r), should be more appropriately written as $C(y, \mu^*)$.

Furthermore, and it is theoretically more relevant, there is a one-to-one relationship between μ^* as a stochastic variable, and its Certainty Equivalence $\rho \equiv Z^{-1}EZ(\mu^*) = \hat{\rho}(p_s)$. The optimum stochastic distribution μ^* and its Certainty Equivalence $\rho = \hat{\rho}(p_s)$ being simultaneously defined by an agent with the NM function Z, the flow consumption-saving decision of this agent, given the distribution of μ^*, should be specified as the consumption-saving decision of the same agent with the Certainty Equivalent rate of return (ρ). These considerations suggest that the consumption demand should be written as $C(y, \rho)$ rather than C(y, r).[6]

Next we deal with the investment function by using the Modigliani-Miller theorem:

Consider V, the nominal operating profit expected by an unlevered firm. V is assumed as a stochastic variable the parameters of which are known by the manager of the firm[7]. Denoting by p_sS the gross value of the firm, where S is the number of the shares already issued, we now define θ such that:

$$\theta \equiv \frac{E(V)}{p_s S}.$$

θ, as defined above, is the "cut-off point" of investment as introduced by Modigliani-Miller[1958]. No investment project whose rate of return is below θ will be accepted by the firm, because the value of the firm will not be maximized if such an investment project is to be accepted. If θ increases, and ceteris paribus, the investment demand will decline, because the number of investment projects whose rate of return is more than θ will diminish. Then, the investment demand is a decreasing function in terms of θ. Because the distribution of V is given to the firm, the investment demand is homogeneous of degree zero with respect to p_s and E(V), and the investment demand, ceteris paribus, is an increasing function in terms of p_s[8].

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6 A more comprehensive formulation of the consumption function, using the permanent income hypothesis and assuming the uncertainty of the future income stream as well as the uncertainty of r, is postponed to section 5.

7 This assumption presupposes that the investment decision is made by the manager under uncertainty. Of course, the manager is treated as a different agent from the consumer.

8 This statement concerns the unlevered firm. According to the first proposition by Modigliani-Miller, op.cit., the

The IS-LM equations (1) and (2) must now be written (more appropriately) as:

$$y = C(y, \rho) + I\left(\frac{E(V)}{p_s S}\right) \cdots\cdots\cdots\cdots \quad (5)$$

$$\frac{M}{p} = M^d(y, \rho) \cdots\cdots\cdots\cdots \quad (6).$$

The model (5) and (6) has two equations with four endogenous variables y, ρ, p and p_s. The number of the endogenous variables has increased from that of the model (1) and (2).

With these preparations, let us now address the missing equation difficulty. The next section will deal with this difficulty in the Classical Model.

3. Certainty Equivalence and the Missing Equation in the Classical Model

The conventional "Classical Model" has been interpreted as composed by the equations (1), (2) and the additional equations (3-2), (3-3) and (3-4). The five equations are a closed system that simultaneously determine y, r, p, ω and L.

Once we notice explicitly that r is not deterministic but stochastic, and specify the notations of each behavioral equations more appropriately as in (5) and (6), it is no longer possible to obtain a closed system by simply adding the equations (3-2), (3-3) and (3-4), for we now have two more endogenous variables p_s and ρ. The five-equation system (5), (6), (3-2), (3-3) and (3-4) is under-determinate, for it contains 6 unknowns (ρ, p, p_s, y, ω and L^*). This model, which by any standard is "classical" because of the equations (3-2), (3-3) and (3-4), has a missing equation.

To close the system, we need to introduce another mathematical relationship. It is provided by

$$\rho = \hat{\rho}(p_s) \cdots\cdots\cdots\cdots \quad (7),$$

which is derived from the Certainty Equivalence relationship

$$\rho \equiv Z^{-1}EZ(\mu^*) \cdots\cdots\cdots\cdots \quad (8),$$

where the distribution of μ^* is dependent on p_s.

gross value of a levered firm is the same as that of the unlevered firm, if the distribution of the operating profit V is the same. Therefore, the investment cut-off point θ for the levered firm is the same for the unlevered firm.

The equations (5), (6) and (7), together with (3-2), (3-3) and (3-4), are a closed system, and we shall call it the "Modified Classical Model". It is to be noted that the Modified Classical Model is closed because (7) is added to the essentially classical system composed by (5), (6), (3-2), (3-3) and (3-4). The only difference between the Classical Model and its modified version is that the latter treats the variable r more explicitly as stochastic. However, if one modified this aspect alone of the classical model, without introducing (7), the modified model would have resulted in an under-determinate system. It implies that the missing equation difficulty is a logical problem not only of the Keynesian model but also of the traditional classical model itself.

The equation(7) is the missing equation that supplements the model composed of (5), (6), (3-2), (3-3) and (3-4). The source of uncertainty of our model is the fact that the variable p_s^e is stochastic. Given the parameters of the distribution of p_s^e, the Modified Classical model determines y, p, p_s, ρ, ω, L^*.

The Modified Classical Model describes the full employment equilibrium under uncertainty in which $L^d(\omega)=L^s(\omega)$ is established by the market forces. The GDP of this model is determined by (3-2)(3-3)(3-4), independently of the other equations. However, this "classical dichotomy" derives only from the assumption that the behavioral equations of the labor sector, the labor supply function in particular, is dependent only on ω, an excessive simplification. Theoretically, the labor supply function should also depend on the non-human capital (which further depends on p_s, among others) as well as on the human capital[9]. A further modification of the behavioral functions will then make all the equations of the model mutually interdependent. In the modified classical model, then, the dichotomy between the monetary and the real sectors is unwarranted.

Further, even if GDP is assumed independently determined by (3-2), (3-3) and (3-4), the money market equilibrium condition (6) (i.e., the LM curve) only specifies a particular relationship to be satisfied between p and p_s, not a proportionality between p and M. In the modified classical model, then, the quantity theory of money does not generally hold.

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[9] As an example, the labor supply of an agent with a huge amount of financial assets will most probably be smaller than the agents without financial assets.

4. The Modified Model in the Keynesian Framework

We now turn to the next question: is it possible to interpret the model (5) and (6) as the Keynesian model?

The equations (5) and (6) are not closed, nor the equations (5), (6) and (7)[10]. Although (5) is usually interpreted as describing the effective demand theory, this interpretation is unwarranted if the system (5), (6) and (7) is underdetermined. If we are to build a macroeconomic model without postulating the wage/price rigidity, we are obliged to introduce the production function as well as the labor market in order to close the model. However, a mere introduction of the equations (3-2)(3-3)(3-4) would simply result in the Modified Classical Model, making it impossible to explain the involuntary unemployment, underemployment equilibrium and other inefficient resource allocations.

This impasse results from the traditional Keynesian model itself. Not giving much theoretical importance to the production function, the traditional Keynesian model determines output by the effective demand alone. Occasionally a linear production function (according to which the output and the labor input are globally proportionate) has been assumed, but it may possibly be inconsistent with the profit maximization behavior of competitive firms[11].

The production function as assumed by (3-3) and (3-4) usually postulates that the marginal product of labor declines continuously. Let us change the assumption slightly and assume instead that the aggregate production function locally contains a portion along which the marginal product of labor is constant. As we shall show below, we will then be able to construct a closed Keynesian model based on the equations (5), (6) and (7), with appropriate behavioral functions replacing the equations (3-2), (3-3) and (3-4).

It is shown in the Appendix that the short run aggregate production function might

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[10] The three equations (5), (6) and (7) must determine p, p_s, y, ρ.

[11] Suppose the production function is globally linear with respect to the labor input. This production function might appear allowable if there is enough redundancy of capital stocks. However, if in this case the profit is positive, then it is possible to increase the positive profit equi-proportionately by maintaining the capital/labor ratio constant and increasing the labor input, because the production function is linear homogeneous. This process will not last forever; either the redundant capital stock is exhausted and the "Classical" equilibrium will arrive; or, the perfect competition (which is implicitly assumed) will be violated. If one were to allow a possibility in which a) the perfect competition is maintained and b) the "Classical" equilibrium does not obtain, there is no choice but to allow, as we shall do, that the zero-profit equilibrium is possible in the short run.

plausibly have a straight portion[12] locally. If the output level is relatively low, it will not be advantageous for an individual firm to operate all the existing capital (considered as the $\overline{K}$ pieces of machines of the same quality). Rather, it will be more advantageous to operate only a part of the machines, keeping the labor/operating-capital ratio constant so that the average labor productivity is maintained constantly at maximum. The effective aggregate production function in such a case contains a straight portion locally.

If we re-specify the production function along this line of thought, the aggregate labor demand curve will have a horizontal portion locally. In Fig.1, the aggregate labor demand curve A'B' is composed by two types of firms I and II. The marginal product of labor of the type I firm is assumed to decline continuously, while that of the type II firm is assumed constant locally. In the interval A'A, where the real wage rate is high, only the firm I will operate. In the interval BB', in contrast, both firms will operate by using all of the existing capital stock (i.e., the classical case).

In the interval AB, where the real wage rate is constant at $\bar{\omega}$, the output of the firm I is constant at a level corresponding to A. However, the optimum demand for labor for the firm II, when the real wage rate is $\bar{\omega}$, is not unique. If the firm II operates at all in the interval AB, the entire value added it produces must be paid out for the wage payment. Apart from the fixed cost, the profit of the firm II will be zero. Unless the exit decision is immediately taken, it is indifferent from the standpoint of the firm II whether to produce a certain quantity of product or not to produce at all[13]. The optimum output in this case is left undetermined, and hence the optimum demand for labor.

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[12] See Appendix for details. We essentially assume that if the output level is relatively low, an attempt to operate all the existing machines simultaneously will result in a low productivity, because each piece of machine is operated by less than efficient number of workers.

[13] The fact that the firm is currently incurring a loss (as much as the fixed cost) does not imply an immediate exit decision. The exit decision making is related to the exit cost including the loss of future profit.

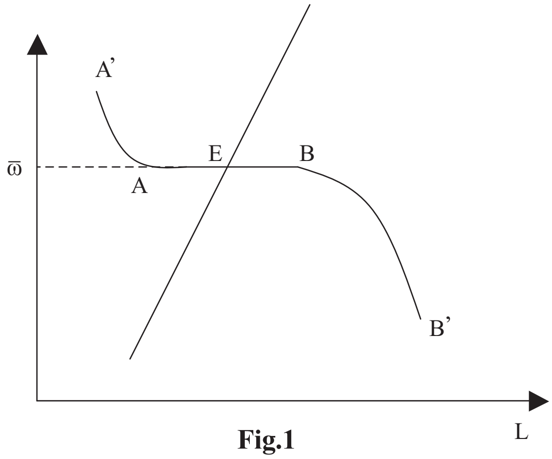

Fig.1

Fig.1 shows that the market adjustment of the real wage, when the labor supply curve and the demand curve meet between A and B, does not warrant the labor market equilibrium at E, the intersection of the two curves. In other words, the aggregate labor demand does not necessarily coincide with the point E through the market adjustment of the real wage rate ω, essentially because the marginal product of labor, though locally, is constant in the neighborhood of E.

The Walrasian market adjustment hypothesis, then, is inapplicable for such a labor market as a theoretical presumption according to which the resource allocation is determined by the adjustment of market prices reflecting the excess demand and supply. One is logically forced to introduce other theoretical presumptions, replacing the Walrasian hypothesis, to determine the allocation of resources (i.e., labor employment).

One possibility is to introduce the Marshallian market adjustment hypothesis, in which case the market equilibrium is established at E. Although E is “inefficient” in the sense that a part of the existing capital stock is not operating, it is “efficient” as long as the labor supply is concerned because E is located on the labor supply curve itself. The Marshallian market adjustment hypothesis does not explain the involuntary unemployment.

We would like to propose that the “Effective Demand Hypothesis”, as a theoretical presumption, is a resource allocation hypothesis applicable to the case where the labor market is characterized by Fig.1. The involuntary unemployment, under this hypothesis, is explained as follows.

Assume that the firm II, having the expectation ($(A^d)^e$) with respect to the demand for its own product, determines the demand for labor required to produce that demand and then to participate in the labor market. The actual GDP, and hence the actual demand for labor, will be determined by $(A^d)^e$. If the actual aggregate demand for labor is located in the interval AE, an involuntary unemployment as well as idle capital stocks will take place. Workers employed by the firm II will be paid more than they desire[14](i.e., although they actually supply labor less than they are willing to offer at the wage rate $\bar{\omega}$, they are paid as much as $\bar{\omega}$).

On the other hand, if the aggregate labor demand is located in the interval EB, workers employed by the firm II will be paid less than they desire (i.e., once they are employed by the firm II, they are obliged to work more than they would be willing to; otherwise they would have to accept unemployment).

The effective demand principle therefore presumes that the actual labor to be supplied by a worker who happened to be employed by the firm II will be determined effectively by the firm[15]. From the "Classical" viewpoint, it is not a natural presumption. However, it is not necessarily so when one describes the resource allocation that takes place in the interval AB of Fig.1. In such a situation, the output level (and the level of employment) of firm I is constant. Apart from the employment opportunity to fill up a vacancy caused by a voluntary retirement from the firm I, unemployed workers may seek job only from the firm II. Any worker who refuses the fact that the terms of labor (labor hours etc.) are effectively decided by the firm II has no choice but to accept unemployment.

To sum up, GDP corresponding to the interval AB where $\omega=\bar{\omega}$ is determined, due to the effective demand principle, at:

$$y=(A^d)^e+(\text{the output of firm I when } \omega=\bar{\omega}) \quad \cdots\cdots\cdots\cdots \quad (9),$$

and the labor employment by the firm II, L^d_{II}, is determined at:

14 Even if the workers currently unemployed may offer to work "cheaply" (at less than the current wage rate $\bar{\omega}$), this will only result in an excess demand in the aggregate labor market, and the market wage rate will be adjusted upward toward $\bar{\omega}$. At $\bar{\omega}$, however, the behavior of the firm II is indeterminate again. Therefore, this attempt by the unemployed workers will not bring the aggregate labor market towards equilibrium.

15 If the workers accept only partially the labor offer determined effectively by the firm II, the firm cannot produce the output based on the demand forecast $(A^d)^e$. Therefore, even if the demand forecast is correct, an unintended inventory variation will take place; in the labor market, the firm II will then reduce the employment offer and an excess supply of labor will develop in the labor market.

$$F_{II}\,(L^d_{II}, \overline{K_{II}}) = (A^d)^e \quad \cdots\cdots\cdots\cdots\cdots \quad (10\text{-}1)$$[16].

Further, the labor employment by the firm I is a value that satisfies:

Marginal product of labor of the firm I $= \bar{\omega}$ $\cdots\cdots\cdots\cdots$ (10-2).

The endogenous variables of the macroeconomic model composed by the equations (5)(6)(7)(9) and (10)[17] are y, ρ, p, p_s, and the aggregate employment (the sum of the employment by the firm I and the firm II). This is the solution when $\omega = \bar{\omega}$. It appears only when the labor supply and labor demand functions intersect in the interval AB of Fig.1 (i.e., when the output level is relatively low). With respect to the missing equation problem, the Certainty Equivalence Method (the equation (7) in particular) is a necessary component to remove the missing equation difficulty of the Keynesian model, though it alone is not enough to remove the difficulty.

5. Uncertainty of the Future Income Stream

We have thus far shown that the missing equation difficulty is solved by introducing the certainty equivalence if there is an uncertainty with respect to the interest rate variable r (i.e., the rate of return on the risky asset).

In this section, we shall deal with another type of uncertainty, i.e., the uncertainty of the future income stream[18]. Needless to say, this problem was analyzed by Hall[1978] and we shall use the permanent income hypothesis as he did. However, unlike Hall, op.cit., we assume that the uncertain future income stream is jointly distributed with the uncertain return from the risky asset. We shall postulate that the consumers will make use of the Certainty Equivalence Method in choosing the optimum inter-temporal consumption plan.

In what follows, we assume for simplicity only two periods. We assume that the current income y is already observed by the consumer, but that the future income y_2 is uncertain. Assuming that y_2 is distributed jointly with r with the joint probability f(r, y_2)[19], we now consider how the consumption function should be modified further.

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16 F_{II}, L^d_{II} and $\overline{K_{II}}$ are the effective production function, labor demand and the capital stock of the firm II respectively.

17 The equations (10-1) and (10-2) are combined, and called (10).

18 See Tanaka-Mutoh[2015] for the case where the future income is uncertain but the return from the financial asset is certain. We show there that the certainty equivalence of the uncertain future income, which reflects the risk preference of the consumer, plays an important role in determining the consumption demand. It is also shown that our approach, which we call Certainty Equivalence Method, is able to deal with any type of risk preference.

19 More precisely, our assumption is that the future income stream is jointly distributed with p^e_s. Because the

With respect to the financial uncertainty, the consumer calculates the Certainty Equivalent rate of return ρ, corresponding to the stochastic rate of return μ ($\equiv\alpha r+(1-\alpha)$ i) from the mixed asset. The optimum share of the risky asset, α^*, is chosen by solving the portfolio maximization problem:

$$\underset{\alpha}{\mathrm{Max}} \int Z(\alpha r+(1-\alpha)i)\, f_r(r)\, dr \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (11),$$

where $f_r(r)$ is the marginal distribution of $f(r, y_2)$ with respect to r.

Once α^* is optimally chosen, the distribution of $\mu^*(=\alpha^* r+(1-\alpha^*)i)$ as well as its certainty equivalence (ρ) is determined. Because the consumer regards ρ (defined as $Z^{-1}EZ(\mu^*)$) and the corresponding stochastic variable μ^* as equivalent, we now assume as the behavioral hypothesis that the consumer regards ρ as the relative price between the present and the future consumption. This amounts to assuming that the consumer regards the present value of the consumption stream using ρ as the discount factor:

$$C+\frac{C_2}{1+\rho}$$

as relevant to the inter-temporal choice of consumption. Defining $B\equiv y+\frac{y_2}{1+\rho}$, the inter-temporal budget constraint is

$$C+\frac{C_2}{1+\rho}= \equiv B^{20},$$

where B is stochastic. The distribution of B is derived from the joint distribution of r and y_2.

The flow-optimization problem for the consumer then is to choose the optimum path (C, C_2), consistent with the certainty equivalent rate of return ρ, and given the distribution of the stochastic wealth variable B. We now use Fig.2 to solve the flow-optimization problem.

Suppose for simplicity that the stochastic variable B has only two possibilities B(a) and B(b), with probabilities π(a) or π(b) respectively. The inter-temporal budget

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distribution of r is related to the distribution of p_s^e by the definition (4), we simplify the notation by assuming the density function with respect to r and y_2 (rather than p_s^e and y_2), keeping in mind that r is also a function in terms of p_s.

[20] Because y_2 stands for the expected future income, B generally stands for the sum of the human and the non-human capital. See Tanaka-Mutoh[2015], Part I for details.

constraint, in terms of Fig.2, also has only two possibilities. Consumer thinks that the inter-temporal budget is either OB(a) or OB(b), with probabilities $\pi(a)$ or $\pi(b)$, but it is not known which of the two actually occurs. How does the consumer choose the optimum consumption path in such an environment?

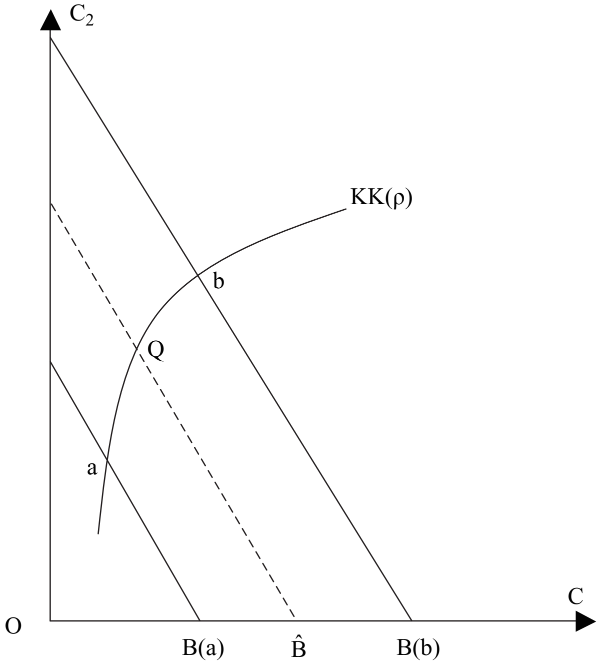

Fig.2

In Fig.2, consider the curve KK(ρ), the income-consumption line corresponding to the rate of return ρ (i.e., the certainty equivalence of μ^*). The point a is the intersection of KK(ρ) with OB(a), and it is the optimum consumption plan if the inter-temporal budget line is B(a). Because B(a) occurs with probability $\pi(a)$, we assume that the consumer will regard the point a as the optimum (and stochastic) consumption plan with probability $\pi(a)$. Likewise, the consumer will regard the point b as the optimum (and stochastic) consumption with probability $\pi(b)$.

We have now established what should appropriately be called the "Optimum Stochastic Consumption Plan", which is represented by the points a and b associated with the probabilities $\pi(a)$ and $\pi(b)$. Notice that we have defined the optimum consumption plan as a stochastic variable, and therefore we are able to consider its expected utility[21].

In order to evaluate the Optimum Stochastic Consumption Plan in terms of expected utility, we need a multi-variable NM utility function U defined over the vectors on KK(ρ). Because we already have Z (a single variable NM function), we will define the function U consistently with Z as follows:

$$U(i) \equiv Z(B(i)),\ i = a,\ b \quad \text{(For the function U, i means the vectors Oa and Ob).}$$

By using U, we will be able to find uniquely a deterministic consumption plan Q on the income consumption line KK(ρ)[22] as the certainty equivalent of the Optimum Stochastic Consumption Plan (i.e, the points a and b). Further, by the correspondence between the functions Z and U, the Certainty Equivalent Consumption Plan Q will exist on the Certainty Equivalence Budget line (shown by the dotted line in Fig.2):

$$C + \frac{C_2}{1+\rho} = \hat{B} \cdots\cdots\cdots\cdots \quad (12)',$$

where $\hat{B}$, the certainty equivalent amount of budget, is calculated as:

$$\hat{B} = Z^{-1}[\pi(a)Z\{OB(a)\} + \pi(b)Z\{OB(b)\}],$$

and $\hat{B}$ is a value that satisfies $OB(a) < \hat{B} < OB(b)$.

The deterministic consumption plan Q is the intersection of KK(ρ) and the Certainty Equivalence Budget Line. The utility of the consumption plan Q is the same as the expected utility of the Optimum Stochastic Consumption Plan, a and b. As we showed elsewhere, the abscissa of the point Q is the optimum current consumption C of the consumer whose risk preference is expressed by the NM utility function Z (or U)[23].

To sum up, the consumption optimization under uncertainty, using the certainty equivalence, is formally and technically reduced to the optimization under certainty. Although the model up to Section 4 has been based on the asymmetric informational assumption with respect to the interest rate and the income variables (i.e., r and y),

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21 Unless the consumption plan is defined as stochastic, it is meaningless to discuss the expected utility of that plan.

22 Q is unique because U is defined only on KK(ρ).

23 From the standpoint of our Certainty Equivalence Method, the inter-temporal consumption plan Q derived by this method is better off than the consumption plan derived by Hall [1978] under the same conditions. The difference between our consumption plan and that of Hall, op.cit. results from the fact that the former satisfies the conventional optimality condition, the equality between the time preference rate and the interest rate variable, while the latter does not satisfy it. For details, see Tanaka-Mutoh[2015], Part I.

the model of this section has assumed that both are stochastic. Under the symmetric informational assumption of Section 5, however, we are allowed to consider that the consumption decision is made as if the informational environment was deterministic. To do so, it is only necessary to evaluate the utility of the uncertain future income stream by using the NM utility function Z, and then to calculate the optimum consumption path under the certainty equivalent budget.

6. Summary

The macroeconomic model composed by (5), (6) and (7), depending on the location of the intersection of the labor demand and the labor supply functions, may generate "Classical" as well as "Keynesian" equilibria. If one adds (3-2)(3-3) and (3-4) to this system, we have the "Modified Classical Model", which describes the economy where the labor supply function in Fig.1 intersects with the labor demand function at a place other than AB in Fig.1. The resource allocation of this equilibrium is arrived at by the price mechanism alone. If on the other hand the labor supply function intersects with the labor demand function in the interval AB of Fig.1, the equilibrium is described by the equations (5), (6), (7), (9) and (10). The resource allocation of this case is arrived at by the effective demand principle, not by the price mechanism.

The macroeconomic model we have developed is in many aspects somewhat "classical"; it assumes an almost traditional production function even when underemployment equilibrium is considered; it assumes optimizing agents; it assumes the perfect competition; and it does not assume any money illusion nor does it assume any wage-price rigidity. Yet the model is capable of generating both full employment and underemployment equilibria. Further, by using the Certainty Equivalence, our model is free from the missing equation difficulty. Whether or not the model is interpreted as Classical or Keynesian, both the general price level and the price of the risky asset are endogenous and simultaneously determined.

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Appendix: On the straight portion of the aggregate production function

This appendix examines the case in which some firms have production functions with locally increasing-returns property. We analyze in particular the relationship between the individual and the aggregate production functions. Throughout the paper, the capital stock is historically given and constant.

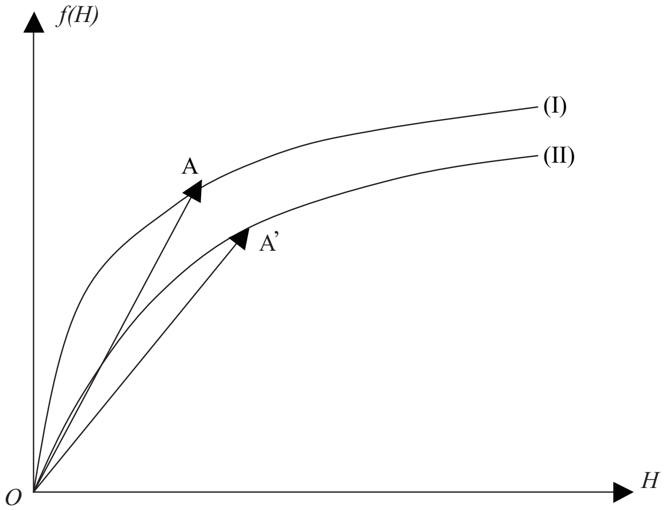

Fig.A-1

Suppose for simplicity two firm types (I, II of Fig.A-1). H is the labor input and f(H) is the output. If the production functions of both firms are well-behaved (see Fig. A-1), the aggregate production function is obtained by (a) taking the points A and A' such that the marginal product of labor at A is equal to the marginal product of labor at A'; and then (b) composing the two vectors OA and OA'. The top of the composite vector shows the relationship between the aggregate labor input and the aggregate output. Parametrically changing A and A', one obtains the aggregate production function, which, by construction, is well-bahaved.

Next we consider the case in which one type of the firms (firm II) has a production function with increasing returns property when the output level is relatively low (Fig. A-2). The increasing-returns property, we assume, appears if the firm II allocates all of the existing capital (i.e., $\overline{K}$ pieces of the machines with the same quality) entirely and simultaneously to the employed labor. In Fig.A-2, we have the heavy line production function with increasing returns between O and H_1. The output shown by the heavy line assumes that all the existing capital stock is operated between O and H_1[24] so that each machine is operated by less than efficient number of labor.

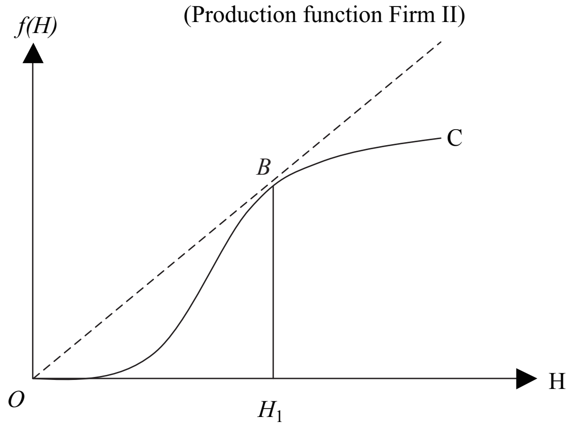

Fig.A-2

If the output is relatively low (i.e., less than BH_1), it is not favorable for the firm II to operate all the machines simultaneously. It is better for the firm to operate only a part of the machines in such a way that each operating machine's productivity is maximized (i.e., $\frac{BH_1}{OH_1}$), leaving the other machines un-operated. The effective production function of the firm II, then, is the dotted line OB in Fig.A-2 as long as the output level is less than BH_1, and the heavy line BC when the output level is relatively high.

Suppose the real wage rate is equal to the slope of the dotted line OB of Fig.A-2. Then the firm II is indifferent between offering a positive employment and offering no employment at all. If, for instance, the demand for the firm II's product is H_1B, it is indifferent from the standpoint of the firm whether to employ as much as OH_1 and produce BH_1, or not to produce at all (and therefore, not to employ at all). In both cases, the profit (apart from the fixed cost) is zero. Unless this firm exits immediately, it is plausible to assume that the firm may decide to produce because the firm may well want to maintain the long run relationship to the customers[25]. If the demand as perceived by

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[24] H_1 is the labor input which corresponds to the tangency of the straight line drawn from the origin with the heavy line production function.

[25] The point B of Fig.A-2 is the "shut-down point", and the short-run supply function is usually interpreted as

the firm II is less than or equal to H_1B, we assume now that the firm will produce as much as the demand.

In FigA-3, the firm I with the well-behaved production function, and the firm II with the locally increasing-returns production function, are depicted. Consider the point (H_1, $f(H_1)$) at which the marginal product of labor of the firm I is equal to the slope of the dotted line OB. If the real wage rate is more than the slope of OB, only the firm I will operate (and therefore, the aggregate production function coincides with the production function (I)).

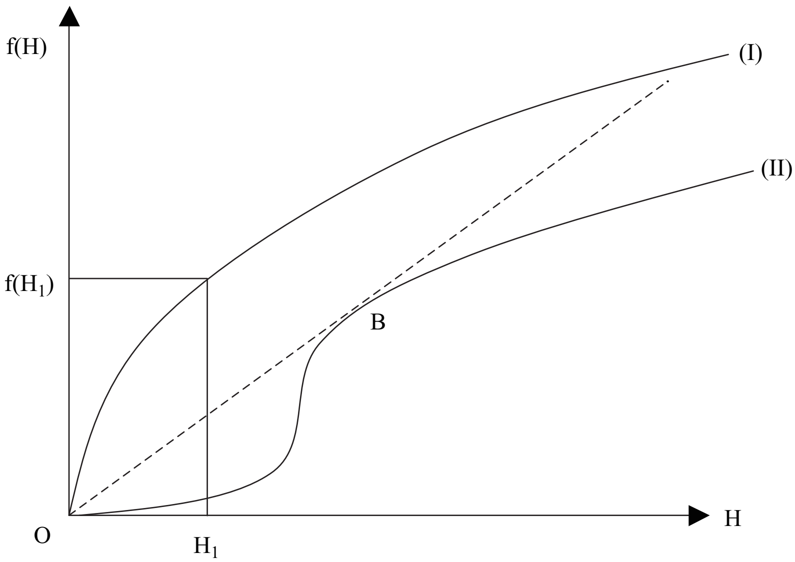

Fig.A-3

When the real wage rate is equal to the slope of the dotted line OB, the output of the firm I is constant at $f(H_1)$. The firm II, on the other hand, produces as much as the demand for its products. Then the aggregate product is equal to "$f(H_1)$+demand for the firm II's product". When the real wage rate is equal to the slope of OB, then, it is possible to assume that the aggregate output y is determined by the aggregate

discontinuous at the output level BH_1. If the firm does not consider the exit as the immediate possibility, however, there is no strong reason to assume that the output discontinuously jumps form BH_1 to zero at the point B. Under our interpretation, the short-run supply curve in this case is multi-valued; corresponding to the real value of the goods (i.e., $1/\omega$), there are multiple output levels between zero and BH_1.

demand for it. In that case, both output and employment offer by the firm I is constant at $f(H_1)$ and OH_1, while the output and the employment offer by the firm II will vary in accordance with the demand for its own products.

The aggregate production function, then, has a straight portion locally at a place where the real wage rate is equal to the slope of the dotted line OB. As long as the market equilibrium takes place on that portion, the output of the firm I is constant at $f(H_1)$ while the output of the firm II will vary along the dotted line OB. The straight line portion of the aggregate production function and its non-straight line portion is smoothly connected, because at the connecting point the marginal product of both firms are equal.

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