Consumption Demand, Uncertainty and Stochastic Process

Takafumi Tanaka, Takahiko Mutoh*

Abstract

Part I of this paper reexamines the consumption demand under uncertain future wage income. In particular, we comprehensively reexamine the "Expected Utility Maximization Method" (EUM, developed by Hall[1978]), and propose an alternative method ("Certainty Equivalence Method", CEM) to derive the consumption demand under uncertainty. We emphasize that the consumption demand should be established as a stochastic variable before considering its expected utility, an important logical point which seems to have escaped appropriate professional attention. We will show that, in contrast to EUM, our method (CEM) is applicable to any type of risk preference.

Based on Part I, we examine in Part II how to derive a refutable hypothesis concerning the consumption demand as a stochastic process, consistently with the consumption demand derived by CEM. Again, it is necessary to reconsider the problem comprehensively, particularly in relation to the risk preference of the consumer. It will be shown that the risk neutrality is one of the important sufficient conditions for the consumption time series to have the martingale property.

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Introduction

This paper comprehensively reexamines the consumption demand under uncertainty represented by stochastic wage-income path. We divide the paper into two parts : Part I examines the consumption demand when the uncertain wage-income path is given as a (joint) statistical distribution; Part II examines the dynamic environment in which the distribution of the wage-income path changes as historical time develops, and we shall analyze the time series property of consumption demand as a stochastic process.

In part I, two basic but conceptually different problems must be solved : one is (a) to optimize the inter-temporal consumption path; and another is (b) to reflect consumer's risk preference over the optimized consumption path.

Since Robert Hall's famous paper (Hall[1978]), the "Expected Utility Maximization Method (EUM)" has been used as the standard technique as well as the basic behavioral assumption to solve the two problems. As we shall see below, however, EUM has several shortcomings including the fact, particularly conspicuous when the optimization target is additively separable, that only the risk averting consumer is possible to analyze. The more basic shortcoming, however, is that EUM tends to mix up the conceptually different problems (a) and (b) in attempting to solve them simultaneously; the economic logic behind this method, concerning in particular the role of risk preference in the consumption choice, is by no means clear as it might be widely recognized.

In an attempt to rectify these shortcomings, we propose what we shall call the "Certainty Equivalence Method (CEM)" as more appropriate to solve the problem at hand. We will show that the problems (a) and (b) are to be solved consecutively but logically separately as the First and the Second Steps. In the First Step, the problem (a) is to be solved, though not necessarily by applying a von Neumann–Morgenstern (NM) type expected cardinal utility. As we shall see, there is no presumption that the problem (a) must be solved as the Expected Utility Maximization problem. Rather, the problem to be solved in the First Step is to define the optimum consumption demand as a stochastic variable, a task which is to be solved by maximizing a target function (not necessarily of NM type) with respect to each and every state of the stochastic budget. In our view, it is logically inappropriate in the First Step to consider the expected utility of consumption demand, for the consumption demand in that step is not yet defined as stochastic at all.

Once the First Step is completed, we may consider its expected utility, which we shall do in the Second Step by introducing the NM utility function enabling us to consider the risk preference. In particular, we shall solve the problem (b) in the Second Step by considering the Certainty Equivalence, and use the NM utility function for that purpose. By making use of "Certainty Equivalence", the relationship between the consumer's risk preference and the flow consumption demand is economically much more clear under CEM than under EUM. Further, CEM makes it possible to deal with any type of risk preference.

The analysis of Part I is the basis of analyzing the consumption demand as a stochastic process, the subject of Part II. Again, it is a field of research initiated by Hall, and it has attracted wide attention[1] owing probably to a well-known statement that "consumption is a random walk apart from trend[2]". Given that CEM differs from EUM, it is necessary to examine the property of consumption as a stochastic process under CEM.

In Part II, we shall focus on the dynamic behavior of consumer who observes the time series data of the market wage-income (treated as exogenous and stochastic), and then uses that information to compose the subjective stochastic distribution of human capital, the discounted sum of the future wage-income path. Some behavioral hypotheses will be proposed, consistently with the mathematical treatment of stochastic process, to describe the way how the consumer composes the subjective stochastic distribution. Only after this somewhat roundabout step, we may reasonably consider the property of the consumption time series as a stochastic process.

With respect to the consumption stochastic process, we shall show sufficient conditions under which the consumption may show the martingale (or random walk) property. These conditions will turn out as more rigid than are usually recognized; they include, in particular, the risk neutral consumer with a homothetic utility function to compose the inter-temporal consumption path. A specific informational environment, to be defined as "Stochastic Perfect Foresight", is also crucial.

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1 See for instance Flavin[1981], Cambell-Deaton[1989], Deaton[1987], Epstein-Zin[1989], Flavin[1993], Kreps-Porteus[1978], Selden[1978] and Quah[1990]. See also Romer[2012], Attanasio[1999] and Blanchard-Fischer[1989] for the relevant articles.

2 Hall[1978], p.975.

Part I : Consumption Demand under Uncertainty

I-1 : The Model : Basic Notations

At time T, the consumer observes ω_T, the market wage-income for period T, and optimally chooses the inter-temporal consumption plan vector

$$\overrightarrow{C_T} \equiv (C_T, C_T^{T+1}, \ldots, C_T^{T+J}, \ldots).$$

C_T is the consumption demand for period T, while C_T^{T+J}, $J \geq 1$ is the consumption demand for period T+J as planned at time T[3].

The wage-income W and the accrual from the non-human wealth N compose the consumer's current income. We assume that the consumer, after observing ω_T, composes the joint probability function

$$f_T(\overrightarrow{W_T}) \equiv f_T(W_T^{T+1}, W_T^{T+2}, \ldots, W_T^{T+J}, \ldots) \cdots\cdots\cdots \quad (I : 1).$$

The variable $\overrightarrow{W_T} \equiv (W_T^{T+1}, W_T^{T+2}, \ldots, W_T^{T+J}, \ldots)$ is a stochastic vector whose J-th element W_T^{T+J} $(J \geq 1)$ is the wage-income of the period T+J[4], as expected by the consumer who has observed ω_T (and other information available) at the beginning of period T.

Notice that the uncertainty of the market wage-income, ω_T, is the basic source of uncertainty of the model. The consumer interprets that uncertainty into the stochastic distribution (I : 1) of $\overrightarrow{W_T}$, a stochastic vector which shows the consumer's subjective forecast of the future wage-income path. Further to be noticed is that the uncertainty of consumption demand, to be discussed below, must be derived endogenously from, and consistently with, the uncertainty of the wage-income path as assumed by (I : 1), for the wage-income stream and the consumption stream are interrelated by the budget constraint.

We define the human capital at period T, denoted by H_T, as :

$$H_T \equiv \omega_T + \frac{1}{1+r} H_T^{T+1} \cdots\cdots\cdots \quad (I : 2\text{-}1)$$

$$\text{and } H_T^{T+1} \equiv \sum_{J=1}^{\infty} \frac{W_T^{T+J}}{(1+r)^{J-1}} \cdots\cdots\cdots \quad (I : 2\text{-}2),$$

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3 C_{T+J} and $C_T^{T+J}(j \geq 1)$ should be conceptually distinguished, although they are often mixed up. C_T^{T+J} is the consumption at period T+J, as planned in period T. C_{T+J} $(j \geq 1)$, on the other hand, is the actual consumption in period T+J. The former is the planned future consumption, as planned in period T.

4 The subscript T attached to the variable W_T^{T+J} means that this stochastic variable is composed by the consumer at time T, the physical time. On the other hand, the super script T+J (J = 1, 2, …) means the future time as perceived by the consumer facing the optimization problem at time T. T+J, therefore, is not the physical time per se.

where r is the rate of interest (assumed for simplicity as non-stochastic and constant). The distribution of H_T^{T+1} (the stochastic human capital at period T+1, as expected at period T) is derived from the distribution of $\overrightarrow{W_T}$, with the parameter r. (The variable H_T, with the parameter r, is also stochastic; we may alternatively write it as $H_T(r)$).

Using (I : 2-2), we also define H_T^{T+2}, the human capital at period T+2, as expected at period T. It satisfies :

$$H_T^{T+1} \equiv W_T^{T+1} + \frac{1}{1+r}\left[\sum_{J=1}^{\infty} \frac{W_T^{T+J+1}}{(1+r)^{J-1}}\right] \equiv W_T^{T+1} + \frac{1}{1+r} H_T^{T+2} \quad \cdots\cdots\cdots \quad \text{(I : 2-3)}.$$

Further, we define the wealth (or budget) at the beginning of period T, denoted by B_T, as :

$$B_T \equiv H_T + (1+r)N_{T-1} \quad \cdots\cdots\cdots \quad \text{(I : 2-4)},$$

where N_{T-1} stands for the non-human capital at the beginning of the period T-1.

I-2 : The Certainty Case

Let us start with the certainty case. If the variables W_T^{T+J}, $J \geq 1$ are non-stochastic, then so are H_T and B_T, and the consumer selects the optimum consumption plan out of all the plans that satisfy the budget constraint. In order to select the optimum consumption plan, it is enough if each consumer has the individual preference ordering. Corresponding to B_T, there exists an optimum inter-temporal consumption plan that lies on the income-consumption curve corresponding to the interest rate r. The property of the income-consumption curve itself depends on the preference ordering of the consumer. Using a strictly quasi-concave function V to represent the preference ordering, the optimum consumption plan should satisfy :

$$\frac{\partial V}{\partial C_T^{T+J}} = (1+r)\frac{\partial V}{\partial C_T^{T+J+1}} \quad (J \geq 1) \quad \cdots\cdots\cdots \quad \text{(I : 3-1)}$$

$$\frac{\partial V}{\partial C_T} = (1+r)\frac{\partial V}{\partial C_T^{T+1}} \quad \cdots\cdots\cdots \quad \text{(I : 3-2)}$$

$$C_T + \frac{1}{1+r}\sum_{J=1}^{\infty} \frac{C_T^{T+J}}{(1+r)^{J-1}} = B_T \quad \cdots\cdots\cdots \quad \text{(I : 3-3)}.$$

The equations (I : 3-1) and (I : 3-2) specify how C_T, C_T^{T+1}, C_T^{T+2}, … should develop on the optimum consumption path, i.e., the Euler equation. Combined with the budget constraint (I : 3-3), and given the value of B_T, the optimum dynamic consumption path is determined. Once C_T is determined, and satisfying the following flow identity, the non-human capital of the next period (N_T) is determined :

$$N_T - N_{T-1} \equiv r\, N_{T-1} + \omega_T - C_T \cdots\cdots\cdots \quad \text{(I : 3-4)}.$$

The consumption function implied by (I : 3-1, 2, 3) is generally expressed as

$$C_T = \widetilde{C_T}(r, B_T) \cdots\cdots\cdots \quad \text{(I : 4)}.$$

Similarly, C_T^{T+J}, $J \geq 1$ is a function that depends generally on r and B_T, the latter further depending on $H_T \equiv \omega_T + \frac{1}{1+r} H_T^{T+1}$.

At time T, where H_T, ω_T and N_{T-1} are given, the model determines the optimum inter-temporal consumption plan (C_T, C_T^{T+1}, C_T^{T+2}, …). The equation (I : 3-4) then determines N_T, while (I : 2-4) determines the next period budget B_{T+1}.

I-3 : Consumption as a Stochastic Variable, and the Certainty Equivalence Method

Let us now state our proposal concerning the consumption demand under uncertainty. We now assume that B_T is a stochastic variable the distribution of which, given N_{T-1}, is derived from (I : 2-4) and $f_T(\overrightarrow{W_T})$. We propose the Certainty Equivalence Method (CEM) as the behavioral hypothesis of each consumer. Because CEM is quite different from EUM, we shall compare the two methods in the next section. CEM asserts that the consumption demand under uncertainty is composed by way of the following two steps :

Proposal 1 (First step)

The optimizing consumer knows that the statistical distribution of B_T will occur as $B_T(a)$, $B_T(b)$, …etc with probability $\pi(a)$, $\pi(b)$ and so forth. We shall refer to $B_T(i)$, $i = a$, b, etc., as the "i-th state" of the stochastic variable B_T.

We assume that the consumer at the First Step, corresponding to each and every state of budget $B_T(i)$, will optimize the preference ordering in order to obtain the "state-wise optimum consumption plan $\overrightarrow{C_T}(i)^*$". Because $B_T(i)$ occurs with probability $\pi(i)$, the optimum consumption plan $\overrightarrow{C_T}(i)^*$ itself will be regarded by the consumer as stochastic with probability $\pi(i)$.

Notice that because this optimization is carried out for each "state"of B_T, it requires only the preference ordering as employed in the optimization under certainty. The First Step optimization is carried out exactly the same as the Certainty Case, except that the optimizing conditions as well as the budget constrainT (I : 3-1)~(I : 3-3) are defined in terms of each "state".

The optimum consumption plan vector $\overrightarrow{C_T^*} \equiv (C_T^*, C_T^{T+1\,*}, C_T^{T+2\,*}\ldots)$, derived by this procedure, is a stochastic vector whose each "state" is expressed as $\overrightarrow{C_T^*}(i)$, i = a, b, … Each "state" value vector of $\overrightarrow{C_T^*}(i)$ (including its first element $C_T^*(i)$) represents the optimum consumption plan for the "i-th state budget", $B_T(i)$[5]. The optimum stochastic vector $\overrightarrow{C_T^*}$ is distributed over the income-consumption curve corresponding to the interest rate r.

The logical implication of the First Step is that the uncertainty of our model derives only from B_T, not from the consumption plan as such. As a logical matter, the uncertainty of the consumption plan should be endogenously defined, within the model, by the First Step. Nevertheless, practically all of the existing literature starts the analysis by maximizing the expected utility of consumption[6]. How can one consider, and maximize, the expected utility of a consumption plan before somehow defining the consumption plan itself as stochastic? A logical flaw may be involved in such a procedure, and the existing literature tends to neglect it.

Now that we have composed the optimum $\overrightarrow{C_T^*}$ as stochastic, its first element C_T^* is also composed as stochastic. This means, however, that the consumption demand at period T is not yet determinate. What has been carried out in the First Step is merely to compose the "optimum stochastic" consumption vector $\overrightarrow{C_T^*}$, the optimality being with respect to each "state of budget".

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[5] As a formal matter, it is possible to introduce a strictly quasi-concave utility function V, the value of V being interpreted as cardinal, to carry out the optimization of the First Step. However, what is needed for the First Step optimization is the preference ordering implied by V, not the cardinally numerical value of V as such. The numerical value of V is relevant to the First Step only in so far as it represents the preference ordering. Although it is possible in the First Step to regard the numerical value of V as cardinal, this maximization result depends only on the value of V regarded as the ordinal number.

[6] To quote some of the literature, Epstein-Zin[1989, p.940, 941], drawing on Kreps-Porteus[1978], starts their analysis by assuming the space of consumption lotteries each of which is endowed with some probability density. Selden[1978], similarly, begins the analysis by assuming the stochastic distribution of future consumption. Kreps-Porteus, op.cit. (p.188, Fig.1) shows a probability tree each branch of which is endowed with probability density, without mentioning how each probability value is assigned. In short, the stochastic consumption space is treated as given, followed by the choice of the best consumption lottery by the consumer.
We in contrast think that a) "the space of consumption lotteries" should be treated not as given but as something which is specified by each economic agent, in accordance with the stochastic version of (I : 3-3), the temporary budget constraint; and that b) it is the probability distribution of the latter (i.e., the stochastic version of the temporary budget constraint, i.e. (1-3-3)' below) that can be treated as given.

In order to choose the consumption demand of period T as a determinate value, one must know how the consumer subjectively evaluates the optimum stochastic consumption plan $\overrightarrow{C_T^*}$. It is for this purpose that the consumer's subjective evaluation of uncertainty is introduced in the Second Step. The Second Step of our proposal is as follows :

Proposal 2 (Second step)

Having composed the "optimum stochastic consumption plan $\overrightarrow{C_T^*}$", the consumer is now able to choose a deterministic consumption demand by applying the "Certainty Equivalence Method (CEM)". In order to do so, a function to evaluate the utility of the stochastic vector $\overrightarrow{C_T^*}$ is necessary. It is for this purpose that the von-Neumann – Morgenstern utility function (NM function, hereafter) is introduced in the Second Step. The role of the NM function is to evaluate the cardinal utility of the optimum stochastic consumption plan vector ($\overrightarrow{C_T^*}$) derived in the First Step.

Notice that the optimum vector $\overrightarrow{C_T^*}$, derived in the First Step is distributed only over the income-consumption curve corresponding to the interest rate r, not over the entire inter-temporal consumption space. It is suggested then that the role of the NM function in the Second Step is not to evaluate any of the consumption plans over the entire consumption space, but to evaluate the expected utility of the optimum consumption plan over the income-consumption curve alone. In this respect, the role of the NM function is different from that of the quasi-concave function V in the First Step, which is defined over the entire inter-temporal consumption space. Hereafter, we use the symbol U to represent the NM function employed in the Second Step.

As explained below, one may use U to define uniquely the "Certainty Equivalence Consumption plan $\overrightarrow{\widehat{C_T}}$ (a deterministic vector) corresponding to the "optimum stochastic consumption plan $\overrightarrow{C_T^*}$". By definition the utility of $\overrightarrow{\widehat{C_T}}$ and the expected utility of $\overrightarrow{C_T^*}$, as calculated by U, are the same. We propose to regard $\widehat{C_T}$, the first element of $\overrightarrow{\widehat{C_T}}$, as the deterministic consumption demand of the period T.

I-4 : A Diagrammatic Presentation

We have put forward the proposals 1 and 2 as the behavioral hypothesis of the consumer under uncertainty. We now use Fig.1 to explain the two steps we propose. For diagrammatical simplicity, we assume only two periods (T and T+1) in Fig.1, but the analysis is a general one, readily extendable to include more than two periods.

(First Step)

Consider the budget constrainT (I : 3-3) under uncertainty. B_T is stochastic with r as the parameter, which we write as $B_T(r)$. For simplicity, we assume $B_T(r)$ to have only two "states" $B_T(a; r)$ and $B_T(b; r)$, with probabilities $\pi(a)$, $\pi(b)$, $\pi(a)+\pi(b)=1$. The budget constraints under uncertainty are :

$$B_T(i;r)=C_T+\frac{C_T^{T+1}}{1+r} \quad i=a, b \cdot \cdots\cdots \quad (I : 3\text{-}3)'.$$

In Fig.1, the line B(a) is the budget constraint of the state a. It intersects with the horizontal axis at $(B_T(a; r), 0)$, with the slope $-(1+r)$. Likewise, B(b) is the budget constraint of the state b, with the same slope. It is assumed that $B_T(a; r)<B_T(b; r)$.

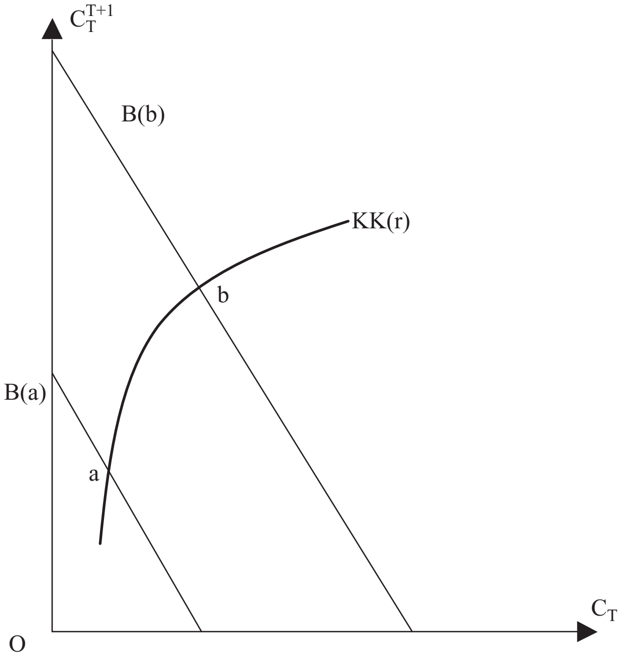

Fig.1

Fig.1 corresponds to (I : 3-2), the optimum condition between C_T and C_T^{T+1}. KK(r) is the income-consumption curve corresponding to the interest rate r. Moving upwards along KK(r) from a to b, inter-temporal consumption plans with successively higher preference ordering will appear. $\overrightarrow{Oa}$ is the inter-temporally optimum consumption plan vector for the budget B(a), because the slope of the budget line is the same as the marginal rate of substitution (MRS) at the point a. Likewise, $\overrightarrow{Ob}$ is the optimum consumption plan if the budget is B(b). The probability with which $\overrightarrow{Oa}$ is optimal is the same as the probability of the budget $B_T(a; r)$ to occur, i.e., π(a). Likewise, $\overrightarrow{Ob}$ is the optimal consumption plan with probability π(b). Further, the MRS of consumption, both at a and at b, is equal to the slope of the budget line, 1+r.

What is the purpose of the First Step? It is merely to define the optimum consumption plan as a stochastic variable. Unlike the past literature, we do not start our analysis by "maximizing the expected utility of consumption", simply because the expected utility of consumption is calculable only after the consumption is somehow defined as a stochastic variable.

To define the consumption as a stochastic variable, the First Step optimization has been carried out as the "state-by-state optimization". Each and every result of the First Step optimization concerns a specific state of budget such as $B_T(a; r)$, other states (such as the budget line B(b)) being irrelevant to the optimization under the state a. Because the First Step optimization is carried out "separately state-by-state," the optimization result for each state is invariant whether the optimization target is V or $\phi(V)$, $\phi'>0$.[7]

Notice that the First Step optimization does not reflect the consumer's risk preference; it must be so, because the transform $\Phi(V)$, $\phi'>0$ generally changes the risk preference (if V is a NM function and unless Φ is linear or affine). Yet the First Step optimization result is invariant whether V or $\Phi(V)$, $\phi'>0$ is employed as the optimization target. Then, the First Step optimization cannot reflect the consumer's risk preference, even if the value of V as such could be interpreted as cardinal; the First Step optimization reflects only the preference ordering information implied by V. It is therefore unnecessary to treat the numerical value of V as cardinal and carry out the First Step optimization as the Expected Utility Maximization (EUM). To do so may not only involve the logical flaw suggested above; it is in fact unnecessary to treat V as cardinal and carry out EUM.

[7] It is important to recognize that even if we use the functions such as (I : 7) or (II : 10a) below --- which are both additively separable NM utility functions --- the First Step optimization requires only the preference ordering implied by these functions.

It must be recognized that the optimum consumption plan $\overrightarrow{C_T^*}$ has been composed by the consumer as a stochastic vector in the First Step. $\overrightarrow{C_T^*}$ is a stochastic vector with two states $\overrightarrow{C_T^*}(a)$ and $\overrightarrow{C_T^*}(b)$, and with probabilities π(a) and π(b) respectively. In Fig.1, $\overrightarrow{C_T^*}(a)$ is expressed as $\overrightarrow{Oa}$, while $\overrightarrow{C_T^*}(b)$ as $\overrightarrow{Ob}$. This completes the First Step.

(Second Step)

Let us now proceed to the Second Step, using Fig.2. We have composed in the First Step the stochastic vector $\overrightarrow{C_T^*}$, each state of which gives the "state-wise optimum consumption plan". C_T^*, the first element of $\overrightarrow{C_T^*}$ is a stochastic variable, too. Because the consumer does not know which state (either a or b) actually occurs, the consumption demand for period T is not determinate as yet. In the Second Step, we introduce U, an appropriate NM function to evaluate the uncertainty of the optimum stochastic consumption plan. By using U, we may uniquely define $\widehat{\overrightarrow{C_T}}$, the certainty equivalence of $\overrightarrow{C_T^*}$, and choose the consumption demand for period T as a deterministic value.

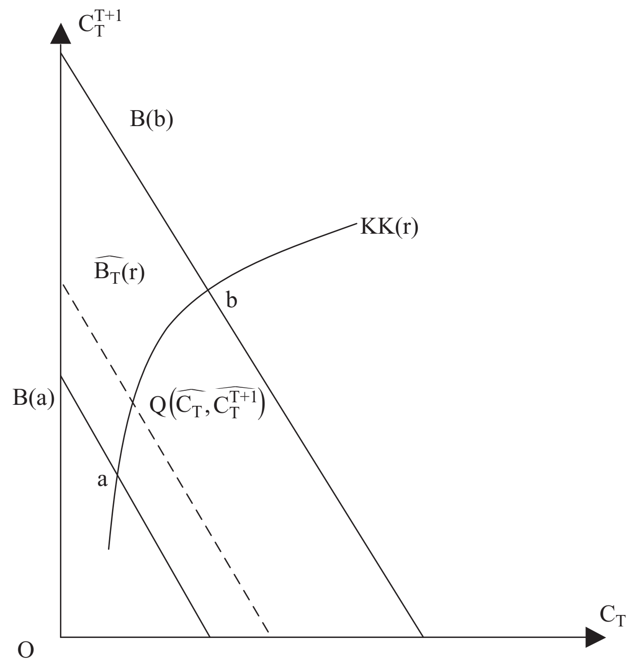

Fig.2

Consider U, a continuous NM utility function, defined over all the consumption plans that lie on KK(r). Notice that U is not defined over the entire C_T - C_T^{T+1} space; it is defined over KK(r) alone, and as such, U must be clearly distinguished from the preference ordering (i.e., the function V) as appeared in the First Step. U is a function giving cardinal evaluation of each and every consumption plan on KK(r), including $\overrightarrow{Oa} \equiv \overrightarrow{C_T^*}(a)$ and $\overrightarrow{Ob} \equiv \overrightarrow{C_T^*}(b)$. We assume U is monotonically increasing if one moves along KK(r) from a towards b. In Fig.2, $U\left(\overrightarrow{C_T^*}(a)\right) < U\left(\overrightarrow{C_T^*}(b)\right)$.

The expected utility of the optimum stochastic consumption plan $\overrightarrow{C_T^*}$, denoted as $\mu\left[U\left(\overrightarrow{C_T^*}\right)\right]$[8], is expressed as :

$$\mu\left[U\left(\overrightarrow{C_T^*}\right)\right] \equiv \pi(a)U\left(\overrightarrow{C_T^*}(a)\right) + \pi(b)U\left(\overrightarrow{C_T^*}(b)\right) \quad \cdots\cdots\cdots\cdots \quad (I:5).$$

Notice that calculating the expected utility of $\overrightarrow{C_T^*}$ as above has no logical flaw suggested earlier, because we have already composed $\overrightarrow{C_T^*}$ as stochastic in the First Step. The value of $\mu\left[U\left(\overrightarrow{C_T^*}\right)\right]$ lies between $U\left(\overrightarrow{C_T^*}(a)\right)$ and $U\left(\overrightarrow{C_T^*}(b)\right)$.

Using $\mu\left[U\left(\overrightarrow{C_T^*}\right)\right]$, we may now uniquely define $\widehat{\overrightarrow{C_T}}$, the certainty equivalence of $\overrightarrow{C_T^*}$, as a determinate vector on KK(r) that satisfies :

$$U\left(\widehat{\overrightarrow{C_T}}\right) = \mu\left[U\left(\overrightarrow{C_T^*}\right)\right] \cdot \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (I:6).$$

With U continuous and monotonically increasing over KK(r), $\widehat{\overrightarrow{C_T}}$ must exist somewhere on KK(r) between the points a and b of Fig.2. We show it by the point Q. The vector $\overrightarrow{OQ}$ is what we now call the "Certainty Equivalence consumption plan $\widehat{\overrightarrow{C_T}}$". Because $\widehat{\overrightarrow{C_T}}$ is not a stochastic but a deterministic vector, its first element is what the consumer may actually demand in period T.

The dotted line of Fig.2 passes Q and is drawn parallel with B(a) and B(b). We call it the "Certainty Equivalence Budget Line". By definition the certainty equivalence consumption plan $\widehat{\overrightarrow{C_T}}$ lies on the Certainty Equivalence Budget Line as well as on the income consumption line KK(r). Then the MRS of consumption at Q is equal to 1+r.

(Risk Preference)

We now consider how the risk preference is related to the consumption demand.

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[8] We use the symbol μ(X) (rather than the symbol E(X)) to express the population mean of the relevant stochastic variable X. The symbol E is reserved for the purpose of expressing the conditional expectation of the relevant stochastic variable X, conditional with respect to a specific σ-algebra. Thus, the expression μ(X) stands for a numerical value, while $E(X|\mathcal{G})$ stands for the $\mathcal{G}$-conditional expectation of the $\mathcal{F}$-measurable stochastic variable X, where $\mathcal{F} \supseteq \mathcal{G}$. $E(X|\mathcal{G})$ is a stochastic variable itself, not a numerical value. Such a distinction of μ and E becomes relevant when one analyzes the consumption as a stochastic process, which is dealt with in Part II.

In general, the risk preference is defined in terms of the relationship between U(μ[X]) and μ[U(X)], where X is a stochastic variable and U is the NM utility function. In the present context, <u>U is defined only over the consumption vectors on KK(r)</u>. We then define a risk averse consumer as the one who satisfies

$$U\left(\overrightarrow{\mu(C_T)}|_{KK}\right) > \mu\left[U\left(\overrightarrow{C_T^*}\right)\right],$$

where we define $\overrightarrow{\mu(C_T)}|_{KK}$ as the consumption plan on KK(r) which satisfies the mean budget $\pi(a)B_T(a; r)+\pi(b)\ B_T(b; r)$. Because $\overrightarrow{\mu(C_T)}|_{KK}$ is defined as a vector on KK(r), its NM utility is able to evaluate by U.

<u>Notice that $\overrightarrow{\mu(C_T)}|_{KK}$ is defined not as the mean vector of the vectors $\overrightarrow{Oa}$ and $\overrightarrow{Ob}$.</u> The latter coincides with $\overrightarrow{\mu(C_T)}|_{KK}$ only when KK(r) is a straight line that passes the origin. As we move along KK(r) from a to b, consumption plans with successively higher preference ordering will appear, and the one that satisfies the mean budget is defined as $\overrightarrow{\mu(C_T)}|_{KK}$.

Applying this definition to (I -6), the risk averse consumer will satisfy :

$$U\left(\overrightarrow{\mu(C_T)}|_{KK}\right) > U\left(\widehat{\overrightarrow{C_T}}\right).$$

Because both $\overrightarrow{\mu(C_T)}|_{KK}$ and $\widehat{\overrightarrow{C_T}}$ lie on KK(r), and because U is monotonically increasing, the budget that satisfies $\overrightarrow{\mu(C_T)}|_{KK}$ is higher than the budget that satisfies $\widehat{\overrightarrow{C_T}}$, if the consumer is risk averse. If Q in Fig.2 is the certainty equivalent consumption plan of a risk averse consumer, then the Certainty Equivalent Budget Line (i.e., the dotted budget line of Fig.2) lies below the mean budget line of the same consumer[9].

Combining the proposals 1 and 2, we propose the following as the behavioral assumption of the consumer under uncertainty :

"Consumer under uncertainty composes, in the First Step, the optimum stochastic consumption plan vector $\overrightarrow{C_T^*}$, and then (in the Second Step, and using the NM function U) composes the certainty equivalent consumption plan $\widehat{\overrightarrow{C_T}}$, corresponding to $\overrightarrow{C_T^*}$[10]. The consumption demand for period T is chosen as the first element of the vector $\widehat{\overrightarrow{C_T}}$".

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[9] For a risk neutral consumer, the certainty equivalent consumption plan is the intersection of KK(r) with the mean budget line. For a risk loving consumer, the certainty equivalent consumption plan is the intersection of KK(r) with a budget line above the mean budget line. A risk neutral (averse) agent is more cautious than a risk loving (neutral) agent when determining the consumption demand.

[10] To our knowledge, Friedman[1957] suggested, although very vaguely, the use of certainty equivalence in deriving the consumption demand under uncertainty. See Friedman, op.cit., p.15.

(NM function Z to evaluate the uncertain budget)

We call this proposal the "Certainty Equivalence Method (CEM)", and compare it in the next section with EUM. Before doing so, however, we show a less complicated way to calculate the certainty equivalent consumption plan $\widehat{\overrightarrow{C_T}}$.

The CEM above has used the NM function U to evaluate the uncertainty of the optimum stochastic consumption plan vector $\overrightarrow{C_T^*}$. Being a function of a vector with many elements, U is a multi-variable function. Notice now that a single-variable NM function Z, which evaluates the uncertainty of the stochastic budget B_T, can be defined consistently with U by the following procedure.

U has been defined with respect only to the consumption vectors on KK(r). Considering the duality of the First Step optimization, there is a one-to-one correspondence between

(1) Human capital in the state i　(H_T(i)),

(2) Budget in the state i　(B_T(i;r)),

and

(3) The optimum consumption plan in the state i　$\left(\overrightarrow{C_T^*}(i)\right)$.

Using the correspondence, and consistently with U, we may define another NM utility function Z to evaluate the utility of B_T(i) as

$$Z(B_T(i;r)) \equiv U\left(\overrightarrow{C_T^*}(i)\right).$$

Thus defined, the function Z is interpreted as a single-variable NM utility function to evaluate the uncertainty of the stochastic variable B_T.

If $U\left(\overrightarrow{C_T^*}(a)\right) < U\left(\overrightarrow{C_T^*}(b)\right)$, the correspondence above implies $B_T(a;\, r) < B_T(b;\, r)$. Then the function Z is monotonically increasing with respect to the increasing budget; and if U is risk averse, then Z is risk averse (noticing that U and Z take the same value with respect to each budget state and the optimum consumption plan corresponding to that budget state).

Using the stochastic variable B_T(r), the certainty equivalent budget $\widehat{B_T}$(r) is now defined as a deterministic budget which satisfies :

$$Z\left(\widehat{B_T}(r)\right) = \mu[Z(B_T(r))].$$

Because Z is a single-variable monotonically increasing function, $\widehat{B_T}$(r) can be written uniquely as :

$$\widehat{B_T}(r) = Z^{-1}\ \mu[Z(B_T(r))].$$

Although the certainty equivalence consumption plan $\overrightarrow{\widehat{C_T}}$ may be calculated by the procedure already explained, it is also possible to calculate it by using Z. Instead of starting the Second Step by postulating U, we may as well start the Second Step by postulating Z. If we do so, we use Z to calculate $\widehat{B_T}(r)$. The Certainty Equivalence Budget Line which we used in Fig.2 is mathematically defined as :

$$\widehat{B_T}(r) = C_T + \frac{C_T^{T+1}}{1+r} \quad \cdots\cdots\cdots\cdots \quad \text{(I : 3-3)''}.$$

$\overrightarrow{\widehat{C_T}}$ is then calculated as the intersection of KK(r) with (I : 3-3)''.

I-5 : Our Proposal (CEM) as Compared with Hall's Method (EUM)

This section explains the optimization method adopted by Hall, op.cit.(Expected Utility Maximization, EUM), and compares it with our own (Certainty Equivalence Method, CEM). We will show that CEM is possible to deal with any type of risk preference, while EUM may basically deal only with risk averters, if the target function is specified as additively separable[11]. We will also show that EUM tends to mix up the two different aspects of the problem at hand, namely (a) to optimize the inter-temporal consumption path; and (b) to reflect consumer's risk preference over the optimized consumption path. We will further show that EUM may be interpreted as a special (or, restricted) case of CEM, and compare the welfare implication of the consumption demand under the two methods.

(EUM : Hall's Method)

Let us first examine EUM, i.e, Hall's method. As we have seen, our approach (CEM) requires two steps to obtain C_T as a determinate value. In contrast, EUM requires effectively the First Step alone, although the First Step optimization is carried out differently from CEM.

Hall's consumer is assumed to maximize the expected value of the following additively separable utility function, with δ as the utility discount factor, and u as the

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[11] Most of the relevant articles assume the additively separable utility function as the target function.

one period utility function[12]:

$$F=u(C_T)+\sum_{J=1}^{\infty}\frac{u\left(C_T^{T+J}\right)}{(1+\delta)^J} \quad u'>0,\ u''<0 \cdots\cdots\cdots\cdots\cdots \quad (I:7).$$

His "principal result" (Hall, op.cit. p.974) requires the following equality be satisfied by the optimum consumption demand between C_T and C_T^{T+1}, and we shall hereafter refer to it as "Hall's Theorem" :

$$(1+r)\sum_{i=a,\,b}\left[\pi(i)\,\frac{1}{1+\delta}\,u'\left(C_T^{T+1}(i)\right)\right]=u'(C_T)\cdots\cdots\cdots\cdots \quad (I:8)$$ [13] [14].

In (I : 8), $\pi(i)$ is the probability with which each "state" of B_T, and hence each "state" of the stochastic variable $C_T^{T+1}(i)$, occurs. How is Hall's theorem derived? An elementary derivation of the theorem, which we shall do by comparing CEM with Hall's method, is as follows[15].

First, consider (I : 3-2), the optimum condition between C_T and C_T^{T+1} under certainty :

$$\frac{\partial V}{\partial C_T}=(1+r)\,\frac{\partial V}{\partial C_T^{T+1}}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (I:3\text{-}2).$$

Now suppose that the target function V is replaced by the additively separable function (I : 7), and suppose that the agent is under uncertainty. In Certainty Equivalence Method (CEM), the First Step optimum condition under uncertainty requires, for each and every state of budget, $B_T(i)$,

$$\frac{du}{dC_T(i)}=(1+r)\frac{1}{1+\delta}\,\frac{du}{dC_T^{T+1}(i)}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (I:3\text{-}2)'$$

be satisfied, where $C_T(i)$ and $C_T^{T+1}(i)$ denote the inter-temporal consumption allocation between T and T+1 when the budget is $B_T(i)$ (occurring with the probability $\pi(i)$). Multiplying both sides by $\pi(i)$ and adding up with respect to all the i' s, we obtain :

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[12] A logical question pertaining to Hall's procedure is this : how the expected utility of consumption can be maximized at the first step when the consumption itself has not yet been defined as stochastic?

[13] Because Hall[1978], p.974 does not distinguish C_T^{T+1} from C_{T+1}, he describes that $\frac{1+r}{1+\delta}\mu[u'(C_{T+1})]=u'(C_T)$ is established at the optimum. As noted before, however, C_T^{T+1} and C_{T+1} are conceptually different. Hall's Theorem should be written as (I : 8).

[14] If $r=\delta$, $T=1$ and if one does not distinguish C_1^2 from C_2, then (I : 8) implies $u'(C_1)=E[u'(C_2)]$. If u is quadratic, further, then an expression $C_1=E[C_2]$ is derived, and this last expression has been often associated with some kind of "Certainty Equivalence". We find it difficult to understand this terminology. The textbook definition of the certainty equivalence is what has been used above in section I-4, and it has nothing to do with the relationship $C_1=E[C_2]$ derived from the quadraticity of u.

[15] To see Hall's original derivation, see Hall, op.cit., Appendix 1 (Theorem).

$$(1+r)\sum_{i=a,\,b}\left[\pi(i)\,\frac{1}{1+\delta}\,u'\left(C_T^{T+1}(i)\right)\right]=\mu[u'(C_T)] \quad \cdots\cdots\cdots\cdots \quad (I:8)'.$$

Let us compare (I : 8)' with Hall's theorem (I : 8). They look quite similar, but yet clearly different because the right hand sides are not the same. In Hall's theorem (i.e., EUM), RHS is $u'(C_T)$, while the First Step optimum condition under CEM requires RHS be equal to $\mu[u'(C_T)]$.

The difference between EUM and CEM, however, disappears if the First Step maximization under CEM is carried out not only under the state-wise budget constraints but also under an additional constraint

$$C_T(i)=\text{constant for any } i \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (I:9),$$

for $\mu[u'(C_T)]=u'(C_T)$ in this particular case.

The constraint (1 : 9) postulates that Hall's consumer chooses the optimum C_T irrespective of the state of B_T (in other words, the consumer composes the optimum "distribution" of C_T as a single point). This postulate is an additional behavioral hyopothesis which our method (CEM) precludes. In a way, EUM and CEM are the same except that EUM implicitly imposes not only the state-wise budget constraints but also the additional constraint (I : 9).

We are now able to characterize Hall's optimization procedure as essentially consisting of two stages : (i) In the first stage, the optimum inter-temporal consumption plan $\overrightarrow{C_T^{T+J}}$, $J\geq 1$ is composed as a stochastic vector, while treating C_T parametrically (Because C_T is treated as parameter at this stage, the maximized expected utility is a function in terms of C_T); and (ii) in the second stage, the optimum value of C_T that maximimizes the expected utility of consumption path (C_T, $\overrightarrow{C_T^{T+J}}$) ($J\geq 1$) is found. Treating C_T parametrically at the first stage means that Hall's optimization is carried out under the additional constraint (I : 9), which is the essential difference between EUM and ECM[16].

In Hall's procedure, the two stages (i) and (ii) are apparently carried out

[16] Although we stated above that the difference between the two methods is shown by the difference of the RHS of (I : 8) and (I : 8)', this distinction applies only when the target function is additively separable. If the target function is more general, the difference of the RHS of (I : 8) and (I : 8)' disappears unless (1 : 9) is additionally assumed. Therefore, the essential difference between the two methods is the additional constraint (I : 9).

simultaneously. His method, therefore, requires virtually the First Step of CEM alone, although the First Step itself is subdivided into two stages. The first stage (i) essentially concerns the optimization of the the inter-temporal consumption path with respect to C_T^{T+1}, C_T^{T+2}, …. The optimization with respect to C_T is not carried out in the first stage in which C_T itself is treated as a parameter. The first stage optimization is carried out by equating the marginal rate of substitution between C_T^{T+J} and C_T^{T+J+1} (for $J \geq 1$) with 1+r, treating C_T as parameter. It is only in the second stage that C_T is optimized. The optimum numerical value of C_T is chosen as satisfying (I : 8), Hall's Theorem.

These two stages, however, are indistinguishably connected in EUM. Further, because Hall's method does not have the Second Step of CEM, it is by no means clear how Hall's optimized consumption plan reflects the consumer's risk preference. Apparently, the risk preference has been treated somewhere in the second stage, but it is not clear how the optimum saving-consumption plan under EUM reflects the risk preference. We find it difficult to understand the economic logic behind EUM.

(Hall's Theorem as Compared with CEM)

Let us summarize some of the features of EUM, i.e., Hall's Theorem (I : 8), comparing it with our method CEM.

Firstly, we note that the result of the First Step of CEM, i.e., the equation (I : 8)', corresponds to the Euler equation in the Certainty case, requiring MRS between C_T and C_T^{T+1} [17] (to be denoted as MRS(T, T+1)) be equal to 1+r for optimum. The equation (I : 8)' is the uncertainty version of that relationship, obtained first by considering the Euler equation for each and every $B_T(i)$, and then by calculating the mean of all the states. The equation (I : 8)' then presupposes that MRS(T, T+1) = 1+r for each and every state of budget.

In contrast, Hall's Theorem (I : 8) does not satisfy MRS(T, T+1)=1+r for each and every state of budget, because $C_T(i)$ under this theorem must satisfy the additional constraint (I : 9) irrespective of the state of budget. Because of this requirement, Hall's theorem is not consistent with the condition MRS(T, T+1)=1+r for each and every state

[17] If the utility function is specified as (I : 7), an additively separable function, then MRS between C_T and C_T^{T+1} is defined as $\frac{u'(C_T)}{u'(C_T^{T+1})/(1+\delta)}$.

of budget[18].

Secondly, we note that the second stage of Hall's method is carried out by searching a value of C_T that maximizes the expected utility of the consumer. If, as Hall does, the additively separable NM utility function (I : 7) is employed as the optimization target, then EUM is consistent only with risk averting consumers. This is because Hall's optimization in the first stage attempts to maximize the (expected) utility of the uncertain consumption path from period T+1 onwards, treating C_T as parameter. This maximization requires u''(c)<0 as the second order condition for the interior maximum, which further implies that the risk-averting consumer is implicitly assumed[19].

Summary of Part I

We now summarize the feature of our approach to the consumption choice under uncertainty.

We have shown in Part I that the consumption choice under uncertainty, given the distribution of the stochastic wage-income stream vector $\overrightarrow{W_T}$, should be solved by way of the following two steps: in the First Step, the consumer composes the optimum stochastic consumption demand vector $\overrightarrow{C_T^*}$ for each state of the stochastic wealth (B_T) whose distribution endogenously derives from the distribution of the stochastic wage-income vector $\overrightarrow{W_T}$. The optimization at this step is carried out with the "stochastic budget constraint (I : 3-3)' at each state" as the only constraint. The purpose of the First Step is simply to compose the optimum stochastic consumption plan vector $\overrightarrow{C_T^*}$ on the income-consumption line KK(r). For this optimization, only the preference ordering among possible consumption plan vectors has been utilized.

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18 If B_T has only two states (a and b), and if we write Hall's theorem alternatively in terms of MRS, we obtain :

$$1+r=\left[\frac{\pi(a)}{MRS(T, T+1: a)}+\frac{\pi(b)}{MRS(T, T+1: b)}\right]^{-1}$$

as the optimum condition, where MRS(T, T+1 : i) means the MRS between C_T and C_T^{T+1} for each state of budget. This condition cannot be equivalent to 1+r=MRS unless MRS(T, T+1 : a)=MRS(T, T+1 : b). From the standpoint of CEM, then, it may not be able to regard Hall's consumer as maximizing the welfare of intertemporal consumption plan.

19 If the target function is additively separable as in (I : 7), then $MRS(T+J, T+J+1)=\frac{u'(C_{T+J})}{u'(C_T^{T+J})/(1+\delta)}$. In order for the first stage interior maximum, it must be true that MRS is decressing, which requires, if the target function is additively separable, that u''(c)<0. It is probably in this context that the recent saving theory tends to put more emphasis on the magnitude of the third derivative of NM utility function to explain the saving. See Kimball[1990] in this context, and the concept of "prudence". We think, however, that the relationship between risk preference and saving-consumption decision is more naturally explained by the second derivative of NM function, though the third derivative is not irrrelevant.

The problem to be solved at the First Step is to compose (define) the optimum consumption as a stochastic variable, not to choose the best lottery among many lotteries. The latter problem is to be solved by maximizing the expected utility, but the First Step problem is not so. It is the usual preference ordering alone, not an NM function, that is needed in the First Step.

The NM utility function U is introduced only at the Second Step. Its role is to evaluate the expected NM utility of the optimum stochastic consumption plan vector $\overrightarrow{C_T}^*$ composed in the First Step. Because $\overrightarrow{C_T}^*$ has been already chosen in the former step, it is enough for the NM function to evaluate the consumption plan $\overrightarrow{C_T}^*$ alone, not the entire consumption plans over the consumption space. By evaluating the NM utility of $\overrightarrow{C_T}^*$, and then using the CEM, we may choose the optimum consumption demand as a deterministic value (at period T). To be sure, this choice explicitly reflects the risk preference implied by the NM utility function.

Notice that the Second Step of our method can be equivalently carried out by introducing a single-variable NM utility function Z, which evaluates the NM utility of the uncertain budget $B_T(r)$. The intersection of the Certainty Equivalence Budget Line (i.e., (I : 3-3)") with KK(r) determines the consumption function under uncertainty, generally expressed as :

$$C_T = \widetilde{C_T}\left(r, \widehat{B_T}(r)\right), \text{ where } \widehat{B_T}(r) \equiv Z^{-1}\,\mu[Z(B_T(r))] \cdots\cdots\cdots \quad (I:10).$$

(I : 10) is the deterministic consumption demand for period T under uncertainty, applicable to both risk averters and risk lovers. It shows explicitly the relationship between the risk preference and the consumption demand. Further, (I : 10) explicitly describes that the consumption demand under uncertainty should be empirically explained, in principle, by the variables that catch not only the first order moment (i.e., mean) but also the higher moments of the stochastic variable $B_T(r)$.

This completes the task of Part I, the fundamental assumption of which has been that the distribution of the stochastic wage-income stream vector $\overrightarrow{W_T} \equiv (W_T^{T+1}, W_T^{T+2}, \ldots, W_T^{T+J}, \ldots)$, and therefore the stochastic distribution of the human capital, is given. We now would like to consider in Part II the time series behavior of the distribution of $\overrightarrow{W_T}$ and its relationship to the time series behavior of the consumption demand. In contrast to Part I, we will deal with the consumption demand as a stochastic process, i.e., the dynamic environment, in Part II.

Part II : Consumption Demand as a Stochastic Process

Introduction to Part II

The analysis of Part I has been restricted to the static environment in which the stochastic distribution of H_T, the human capital at T, is given. In Part II, we deal with dynamic changes in the distribution of H_T as T develops, in order to examine the consumption demand as a stochastic process.

Our consumption function in Part I has been derived rather differently from EUM, and therefore its stochastic process property, the martingale property in particular, needs to be reexamined comprehensively. In doing so, we are obliged to take a roundabout way, examining first how the consumer composes the subjective distribution of H_T. Moreover, because the martingale property is generally defined in relation to a series of expanding σ-algebra, which in turn reflects the accumulation of available information, we must start our analysis by hypothesizing how the consumer revises the stochastic space over which H_T is defined successively.

In what follows, we describe in general terms how the consumer successively revises the stochastic space (Section II-1) as well as the distribution of H_T defined over that space (Section II-2). Only after these roundabout logical steps will it be possible to propose a refutable theoretical hypothesis under which the consumption demand as a stochastic process may possess the martingale property. We shall show, in Sections (II-3) and (II-4), under what sufficient conditions the time series consumption data may show the martingale (or random walk) property. We shall show in particular that this property appears under more rigid restrictions than are usually recognized. In order for consumption to show this property, one must assume a risk neutral consumer with homothetic utility function (to be employed in the First Step optimization of Part I). Further, this result obtains in an informational environment, to be called the “Stochastic Perfect Foresight”.

II-1 : The Basics

The basic assumption is that the market wage–income ω is a stochastic variable the distribution of which is external to the economic behavior of each competitive consumer. The consumer is assumed to have no knowledge of ω's distribution, to be called the “nature”.

What the consumer at T does is to observe the current market wage-income ω_T. After observing ω_T (and other information which the consumer thinks relevant to the future wage path), the consumer subjectively composes the stochastic future path of the wage income. The consumer composes the variable $\overrightarrow{W_T} \equiv (W_T^{T+1}, W_T^{T+2}, \ldots, W_T^{T+J}, \ldots)$ as a stochastic vector whose J-th element W_T^{T+J}, $(J \geq 1)$ is the wage-income of period T+J[20], as expected by the consumer in period T. The symbol T, appearing both as the subscript and the superscript of the variable W_T^{T+J}, means that this stochastic variable is subjectively composed by the consumer at time T, the physical time. On the other hand, the symbol $J(J=1, 2, \ldots)$, appearing in the superscript T+J, means the future time T+J as conceived by the consumer at time T. The symbol T+J, given T, is not the physical time itself. We shall hereafter refer to the physical time T as the “vertical” time. On the other hand, we shall refer to T+J(given T) as the “horizontal” time. The distribution of ω, the “nature”, gives the time series data $\{\omega_T, \omega_{T+1}, \ldots\}$ along the “vertical” time to be observed successively by the consumer, who then revises the subjective stochastic vector $\overrightarrow{W}$ of the future wage income path defined along the “horizontal” time, i.e., the superscript T+J $(J \geq 1)$.

Once the distribution of $\overrightarrow{W_T}$ is subjectively composed, the consumer may calculate the distributions of H_T, H_T^{T+1} and H_T^{T+2}, defined as (I : 2-1), (I : 2-2) and (I : 2-3). Notice that the superscript of H_T^{T+1} (as well as the superscript of H_T^{T+2}) refers to the horizontal time. From the distribution of H_T, the consumer may calculate the distribution of B_T defined as (I : 2-4).

According to our analysis in Part I, the deterministic consumption demand for period T is expressed as :

$$C_T = \widetilde{C_T}(r, \widehat{B_T}) \quad \cdots\cdots \quad \text{(II : 1)}$$

$$\widehat{B_T} \equiv Z^{-1}\,\mu[Z(B_T)] \quad \cdots\cdots \quad \text{(II : 2)},$$

where Z is a von-Neumann-Morgenstern (NM) utility function that evaluates the NM utility of the uncertain variable B_T. The consumption demand C_T depends on the interest rate r and the “Certainty Equivalent Budget $\widehat{B_T}$”, which is dependent on B_T.

[20] We assume, for simplicity, that W_T^{T+J} is discretely distributed with a finite number of occurrences.

The distribution of B_T, and therefore the numerical value of $\widehat{B_T}$, change dynamically as the market wage income is observed by the consumer as ω_T, ω_{T+1}, … etc. At time T+1, and given the information ω_{T+1}(and other information available at T+1), the probability function(I : 1) is revised from $f_T(\overrightarrow{W_T})$ to $f_{T+1}(\overrightarrow{W_{T+1}})$, from which the distribution of H_{T+1} and B_{T+1} are recalculated. Consumption as a stochastic process reflects essentially the dynamic revision of the distribution of human capital, which itself reflects the additional information available at each period. In order to examine the consumption as a stochastic process, we must examine the relationship between the observation series ω_T, ω_{T+1}, …, and the dynamic revision of the distribution of human capital H (and that of B). We now address how H and B are revised successively as the vertical-time develops.

II-2 : Human capital as a stochastic process and the concept of "Stochastic Perfect Foresight"

At this stage, we must take a roundabout way and explicitly propose a behavioral hypothesis concerning how the consumer subjectively considers the "nature", the distribution of the market wage-income ω, which successively occurs as $\{\cdot\cdot, \omega_T, \omega_{T+1}, \omega_{T+2}, \cdot\cdot\}$. We assume that the consumer at the vertical time T observes ω_T (and other relevant information available at T), and then composes the σ-algebra $\mathcal{F}_T$. $\mathcal{F}_T$ may then be called "All the Information available at T."[21] The consumer then composes the subjective stochastic vector $\overrightarrow{W_T} \equiv (W_T^{T+1}, W_T^{T+2}, \ldots W_T^{T+J}, \ldots)$ as $\mathcal{F}_T$- measurable stochastic variables. Each component of $\overrightarrow{W_T}$, say W_T^{T+1}, is a mapping from ω_T to R (i.e., the real number). It is the consumer who determines the property of the mapping subjectively, using the information ω_T(and other relevant information available at T).

This is what we hypothesize as the way how the stochastic vector $\overrightarrow{W_T}$ is composed. The distributions of H_T^{T+1}, H_T^{T+2}, H_T and so forth are calculated from it. The consumer in period T regards the stochastic W_T^{T+1} as the subjective forecast of ω_{T+1}, the wage-income to be observed in the next period, T+1.

Needless to say, there is no presumption to regard the distribution W_T^{T+1} as

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[21] To compose $\mathcal{F}_T$, the consumer must start by composing a partition Ω^T of the sample space Ω. The consumer then composes the σ-algebra $\mathcal{F}_T$ as the smallest σ-algebra which contains the members of Ω^T. For simplicity, we assume that Ω^T is a finite partition of Ω. Because Ω^T is composed by using the information available at time T, and because $\mathcal{F}_T$ is uniquely determined by Ω^T, we may regard $\mathcal{F}_T$ itself as "All the Information available at T".

somewhat "close" to the distribution of ω, the "nature" to occur in the next period as ω_{T+1}. W_T^{T+1} is a subjective distribution with several possible values to occur, while ω_{T+1} is the value to be observed only in period T+1. There is no strong reason for the distributions of ω and W_T^{T+1} to coincide. Further, although it might happen that one of the possible values for W_T^{T+1} to occur actually occurs as ω_{T+1}, there is no presumption that such is always the case. After all, W_T^{T+1} is a statistical distribution composed subjectively by the consumer, and there is no guarantee that the distributions of ω and that of W_T^{T+1} are identical.

Be that as it may, when T+1 arrives, the consumer observes ω_{T+1}as well as the other information available at T+1, and composes $\mathcal{F}_{T+1}$. Then the stochastic variables W_{T+1}^{T+2}, W_{T+1}^{T+3}, …(as well as H_{T+1}^{T+2}) are composed as $\mathcal{F}_{T+1}$- measurable stochastic variables. We assume that $\mathcal{F}_T \subseteq \mathcal{F}_{T+1}$[22], and that the stochastic variable series $\{H_T^{T+1}, H_{T+1}^{T+2}, \ldots.\}$ is composed as adapted with the expanding σ-algebra series $\mathcal{F}_T \subseteq \mathcal{F}_{T+1} \subseteq \ldots$. Essentially, such are what we propose as the behavioral hypothesis of the consumer.

Let us now consider, under this hypothesis, the property of the stochastic variable series $\{H_T, H_{T+1}, \ldots\}$, which essentially determines the property of consumption as a stochastic process. The stochastic variable H_T is defined by (I : 2-1) and (I : 2-2) as

$$H_T \equiv \omega_T + \frac{1}{1+r}\sum_{J=1}^{\infty}\frac{W_T^{T+J}}{(1+r)^{J-1}} \quad \cdots\cdots\cdots\cdots \quad \text{(II : 3)},$$

which, given ω_T (and other information available at T), is composed as $\mathcal{F}_T$- measurable. In order to examine the property of H_T as a stochastic process, we would like to know, essentially, the relationship between the two successive stochastic variables H_{T+1} and H_T, namely;

$$H_{T+1} - (1+r)H_T = -(1+r)\omega_T + (\omega_{T+1} - W_T^{T+1}) + \frac{1}{1+r}[H_{T+1}^{T+2} - H_T^{T+2}] \quad \cdots \quad \text{(II : 4)}.$$

The stochastic difference equation (II : 4) is not generally well-defined, because H_{T+1} and H_T are defined over different σ-algebras. However, our behavioral hypothesis that $\mathcal{F}_T \subseteq \mathcal{F}_{T+1}$ allows us to take the $\mathcal{F}_T$- conditional expectation of (II : 4) to obtain :

$$E[H_{T+1}|\mathcal{F}_T] - H_T = rH_T - (1+r)\,\omega_T - E[\eta_{T+1}|\mathcal{F}_T] \quad \cdots\cdots\cdots \quad \text{(II : 5)},$$ [23]

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[22] This assumption is justified by a more basic behavioral hypothesis according to which some (possibly all) components of Ω^T is subdivided and included by Ω^{T+1}, reflecting the arrival of the new information at T+1. In short, the partition of Ω becomes "finer and finer" as the vertical time proceeds and information accumulates.

[23] Notice that what we mean by the conditional expectation is the generalized one, the conditioning being with respect to the σ-algebra. It is the generalization of the conditional expectation in the elementary sense. The

where we define

$$\eta_{T+1} \equiv \eta_{T+1}(A) + \eta_{T+1}(B) \cdots\cdots \quad (II:6),$$

$$\eta_{T+1}(A) \equiv (\omega_{T+1} - E(W_T^{T+1} | \mathcal{F}_T)) = \omega_{T+1} - W_T^{T+1} \text{ [24]} \cdots\cdots \quad (II:7)$$

$$\eta_{T+1}(B) \equiv \frac{1}{1+r}[E(H_{T+1}^{T+2}|\mathcal{F}_{T+1}) - E(H_T^{T+2}|\mathcal{F}_T)] = \frac{1}{1+r}[H_{T+1}^{T+2} - H_T^{T+2}] \cdots \quad (II:8)\text{[25]}.$$

In these definitions, we note that $\eta_{T+1}(A) \equiv \omega_{T+1} - W_T^{T+1}$ is a stochastic variable well-defined in period T+1, but not in period T because ω_{T+1} is yet to be observed in the future. Likewise, $\eta_{T+1}(B) \equiv \frac{1}{1+r}[H_{T+1}^{T+2} - H_T^{T+2}]$ is not well defined in period T, because H_{T+1}^{T+2} is composed after observing ω_{T+1}. Further, because H_{T+1}^{T+2} and H_T^{T+2} are composed over different σ-algebras, the expression $H_{T+1}^{T+2} - H_T^{T+2}$ in (II : 4) is generally undefined. When this term appears in (II : 5), however, it appears as a well-defined conditional expectation $E[\eta_{T+1}(B)|\mathcal{F}_T]$.

Let us at this stage propose to define the concept of "Stochastic Perfect Foresight", which will turn out in the next section to be one of the crucial conditions under which the consumption as a stochastic process may show the martingale property. We in particular propose to define "Stochastic Perfect Foresight" as a steady state in which (a) the stochastic spaces $\{\Omega, \mathcal{F}_T, P_T\}$ and $\{\Omega, \mathcal{F}_{T+1}, P_{T+1}\}$ coincide, and (b) H_{T+1}^{T+2} and H_T^{T+2} are composed by the consumer as the stochastic variables with the same distribution over the same σ-algebra $\mathcal{F}_T = \mathcal{F}_{T+1}$, so that not only $E[\eta_{T+1}(B)|\mathcal{F}_T]$, but also $\eta_{T+1}(B)$ itself is a stochastic variable the mean of which is zero (i.e., $\mu(\eta_{T+1}(B)) = 0$)[26].

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difference between the conditional expectation in the elementary sense and that in the general sense is as follows. Consider stochastic variables x and y, jointly distributed with the probability f(x, y). The conditional probability of y|x(i.e., the x-conditional distribution of y) in the elementary sense is defined as f(x, y)/g(x), where g(x) is the marginal probability of x. In this elementary definition, the conditioning refers not with respect to the σ-algebra, but with respect to x. It is apparent that the literature concerning the consumption demand under uncertainty, as well as the literature concerning the rational expectation hypothesis in general, has been in most cases developed within the stochastic framework of the conditional expectation in the elementary sense. We use the conditioning in the general sense, for it is more natural to consider the consumption demand as a stochastic process within a framework of the changing (and expanding) σ-algebra.

24 If a stochastic variable X is $\mathcal{F}$-measurable, then $X = E[X|\mathcal{F}]$ holds (For proof, see Cinlar[2010], p.144, Th.1.10, proposition a). The expression $E(W_T^{T+1}|\mathcal{F}_T)$, included in the definition of $\eta_{T+1}(A)$, may then be written simply as W_T^{T+1}, for W_T^{T+1} is $\mathcal{F}_T$-measurable.

25 Because H_{T+1}^{T+2} is $\mathcal{F}_{T+1}$-measurable, $E(H_{T+1}^{T+2}|\mathcal{F}_{T+1}) = H_{T+1}^{T+2}$. Likewise, $E(H_T^{T+2}|\mathcal{F}_T) = H_T^{T+2}$ because H_T^{T+2} is $\mathcal{F}_T$-measurable.

26 As noted above, we distinguish the symbol μ from the symbol E.

We will show in the next section that B_T, the stochastic budget, will have the martingale property under "Stochastic Perfect Foresight", and that this property will be transmitted to the time series property of C_T under several conditions including the risk neutrality of the consumer. Because the "Stochastic Perfect Foresight" concept is importantly related to the time series property of consumption, let us examine its implications in more detail.

The "Stochastic Perfect Foresight" concept has been defined above in association with the term $\eta_{T+1}(B)=\frac{1}{1+r}[H_{T+1}^{T+2}-H_T^{T+2}]$. Assuming that the consumer is currently behaving under "Stochastic Perfect Foresight", let us further examine its implications concerning the term $\eta_{T+1}(A)\equiv\omega_{T+1}-W_T^{T+1}$, the prediction error as a stochastic variable (at T+1).

What would happen if the consumer in period T+1 had regarded ω_{T+1}(and the other information available in T+1) as "innovative" (i.e. "surprising") in the sense that the observed ω_{T+1} was not included in the theoretically possible values of W_T^{T+1} to occur? The consumer would then have revised the stochastic space from$\{\Omega, \mathcal{F}_T, P_T\}$to $\{\Omega, \mathcal{F}_{T+1}, P_{T+1}\}$. Then, all the stochastic variables defined over that space would have been revised, so that H_{T+1}^{T+2} would have been differently distributed from H_T^{T+2}.

The "Stochastic Perfect Foresight" has been defined as a situation under which no such revisions take place. It is then suggested that the market wage income ω_{T+1}, as observed in period T+1, coincides with one of the theoretically possible values of the subjective stochastic variable W_T^{T+1} in the "Stochastic Perfect Foresight".

We now propose to accept, as the inter-temporal (i.e., the "vertical-time") behavioral hypothesis, that the consumer under Stochastic Perfect Foresight regards the observed value ω_{T+1} (and all the other information newly obtained at period T+1) as non-innovative, and composes the stochastic space $\{\Omega, \mathcal{F}_{T+1}, P_{T+1}\}$ exactly the same as the stochastic space $\{\Omega, \mathcal{F}_T, P_T\}$. Then the consumer will compose H_{T+1}^{T+2} as the same stochastic variable ($\mathcal{F}_T$-measurable) as H_T^{T+2}. The term $\eta_{T+1}(B)$ will be a stochastic variable composed as the difference between the two $\mathcal{F}_T$-measurable stochastic variables with the same distribution. The conditional-expectation relationship (II : 8) will of course hold, but the sigma-algebra remains the same as the vertical-time develops under the "Stochastic Perfect Foresight". Moreover, the population mean of $\eta_{T+1}(B)$ (to be written as $\mu(\eta_{T+1}(B))$), will be zero in this case. We will make use of these properties to analyze in the next section the consumption demand as a stochastic process.

Before proceeding to the next section, let us briefly note the conceptual distinction between the "Stochastic Perfect Foresight" and what is generally regarded as the "Rational Expectation (RE)". The rational expectation (RE) should be defined as the case in which ω, the distribution of the market wage-income of the next period, and that of the stochastic variable (W_T^{T+1}) coincide. Although the former is a stochastic distribution whose samples are successively observed along the "vertical" time, the latter is a (subjective) stochastic distribution defined over the "horizontal" time.

It is surely possible to think of a case in which the vertical and the horizontal distributions coincide; in order for the coincidence, however, we need more assumptions with respect to ω than we have hitherto made, as we will discuss below[27]. We assume no such coincidence in defining "Stochastic Perfect Foresight". It is more plausible to regard RE as a special case of the Stochastic Perfect Foresight, in the sense that ω and W_T^{T+1} coincide[28]. In contrast, the two distributions do not coincide under "Stochastic Perfect Foresight".[29] It will be shown below that "Stochastic Perfect Foresight" is more relevant to the martingale property of the consumption time series.

II-3 : B_T and C_T as Stochastic Process

We have thus far taken a roundabout way to introduce explicitly the consumer's behavioral hypothesis, i.e., how the distribution of H_T(as well as B_T) is revised along with informational accumulation as reflected by the successive revision of $\mathcal{F}_T$. Further, we have introduced the concept of Stochastic Perfect Foresight, which will be shown as relevant to derive the consumption demand as a stochastic process.

Supposing that the dynamic development of the stochastic space $\{\Omega, \mathcal{F}_T, P_T\}$ has arrived at the "Stochastic Perfect Foresight", and using CEM as developed in Part I, we shall now examine how the consumption demand $\{C_T\}$ develops as a stochastic

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[27] See (II : 14) below, as well as the footnote attached to it. At this stage, we assume simply that ω is a stochastic variable, nothing more.

[28] Hall, op.cit., as well as Flavin[1981], p.978, the eq.(6), apparently assume RE and consider that the term E $[\eta_{T+1}|\mathcal{F}_T]$ is zero under this hypothesis. It is certainly correct to consider E $[\eta_{T+1}(B)|\mathcal{F}_T]=0$ under RE, for RE is a special case of the Stochastic Perfect Forecast.
With respect to the term $\eta_{T+1}(A)\equiv\omega_{T+1}-W_T^{T+1}$), it is of course true under RE that E $[\omega-W_T^{T+1})]=0$ (because the distributions of ω and W_T^{T+1} coincide under RE). However, ω_{T+1} as appearing in the definition of $\eta_{T+1}(A)$ is the observed value of ω, and ω is a stochastic distribution. Hall's claim above, as long the term $\eta_{T+1}(A)\equiv\ \omega_{T+1}-W_T^{T+1}$ is concerned, is unjustified.

[29] This means that the prediction error, $\eta_{T+1}(A)\equiv\ \omega_{T+1}-W_T^{T+1}$, will not cancel out on the average under "Stochastic Perfect Foresight".

process. We will show that the stochastic process $\{C_T\}$ may possibly show the martingale property under the following set (a)~(c) of conditions : (a) the consumer is risk neutral; (b) and the utility function to be employed in the First Step of the CEM method is homothetic; and c)the consumer behaves in the Stochastic Perfect Foresight. To our knowledge, the relationship between the risk preference and the property of the consumption time series has not been duly recognized by the literature.

Consider (II : 1), the consumption function under uncertainty (notice that the interest rate r is given as constant). It shows that the time series behavior of C_T is determined exclusively by the time series behavior of the Certainty Equivalent Budget $\widehat{B_T}$, which further is exclusively determined by the time series behavior of B_T. We will therefore focus on the series of stochastic variables $\{B_T, B_{T+1}, B_{T+2}, \ldots\}$. We will first show that B_T as a stochastic process will show the martingale property in the Stochastic Perfect Foresight. We will next discuss the relationship between the consumption demand C_T as a stochastic process and B_T as a stochastic process.

Let us examine the property of B_T $(=H_T+(1+r)N_{T-1})$ as a stochastic process. Using (II : 3) as well as the flow identity $N_T - N_{T-1} \equiv r\, N_{T-1} + \omega_T - C_T$, we obtain :

$$B_{T+1} - (1+r)\,B_T = [H_{T+1} - (1+r)H_T] + (1+r)[N_T - (1+r)\,N_{T-1}]$$
$$= -(1+r)\,\omega_T + (\omega_{T+1} - W_T^{T+1}) + \tfrac{1}{1+r}\,[H_{T+1}^{T+2} - H_T^{T+2}] + (1+r)[\omega_T - C_T]$$
$$= [\omega_{T+1} - W_T^{T+1}] - (1+r)\,C_T + \tfrac{1}{1+r}\,[H_{T+1}^{T+2} - H_T^{T+2}].$$

Suppose the consumer is currently in the Stochastic Perfect Foresight in which the stochastic space composed by the consumer remains the same. Then, as discussed above, H_{T+1}^{T+2} and H_T^{T+2} are identically distributed over the same σ-algebra, so that the term $\frac{1}{1+r}\,[H_{T+1}^{T+2} - H_T^{T+2}]$ drops. We may therefore conclude that $B_{T+1}-(1+r)B_T$ is distributed in the Stochastic Perfect Foresight as satisfying :

$$B_{T+1} - (1+r)B_T = [\omega_{T+1} - W_T^{T+1}] - (1+r)C_T \cdots\cdots\cdots\cdots \quad \text{(II : 9a)}.$$

Substituting the consumption function (II : 1) into (II : 9a), and using (II : 2), the inter-temporal behavior of B_T in the Stochastic Perfect Foresight is expressed as :

$$B_{T+1} - B_T = rB_T + [\omega_{T+1} - W_T^{T+1}] - (1+r)\cdot\widetilde{C_T}(r, Z^{-1}\,\mu[Z(B_T)]) \cdots\cdots \quad \text{(II : 9b)}.$$

Let us now examine (II : 9b). From the standpoint of period T+1, all the RHS variables, apart from ω_{T+1}, are predetermined variables the distribution of which has been composed by using the information available in the former period (i.e., period T). If, therefore, the market wage ω is stochastically distributed with $\omega=\mu_{\omega}+\varepsilon$, $\mu(\varepsilon)=0$, generating successively the observed time series data $\{\cdot\cdot\omega_T, \omega_{T+1}, \cdot\cdot\}$ along the "vertical" time, then the stochastic process of B_T itself (along the "vertical" time) is a martingale, apart from trend. If, moreover, if ε is i.i.d., then B_T follows a random walk with trend.

Now that the property of B_T in the Stochastic Perfect Foresight is shown as a martingale with trend, we now turn to the Consumption demand as a stochastic process.

Consider once again the consumption function :

$$C_T=\widetilde{C_T}(r, Z^{-1}\ \mu[Z(B_T)]).$$

The consumption function, mathematically, is a transform from B_T, a stochastic variable, to C_T, the deterministic consumption demand. It is in fact composed by two transforms, first from the stochastic B_T to its certainty equivalence $Z^{-1}\ \mu[Z(B_T)]$, and then from Z^{-1} $\mu[Z(B_T)]$ to the deterministic C_T through the consumption function $\widetilde{C_T}$. Presumably, then, the martingale property of B_T will be transmitted to C_T in the Stochastic Perfect Foresight, if the two transforms are both linear. Let us see if this conjecture is justified.

It is evident that a) if Z is linear (a risk neutral consumer), the transform Z^{-1} $\mu Z=Z^{-1}Z\mu=\mu$, and μ itself is a linear transform. Further, b) if the utility function to be employed in the First Step of the Certainty Equivalence Method as developed in Part I is homothetic, then the income-consumption curve for the given interest rate r is a straight line that passes the origin. In such a case, the level of budget and the level of consumption is proportionate, making the consumption function linear with respect to budget B_T.

Let us therefore assume a risk neutral consumer who maximizes, in the First Step of CEM, the homothetic preference ordering implied by

$$G=(C_T)^{\alpha}+\sum_{J=1}^{\infty}(C_T^{T+J})^{\alpha}(1+\delta)^{-J}\quad(0<\alpha<1)\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\quad(\text{II}:10\text{a})$$[30].

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[30] Because $0<\alpha<1$, it is possible to interpret the function G, an additively separable function, as a cardinal (von Neumann-Morgenstern type) utility function representing a risk averse consumer. It might be conjectured then that this consumer must be treated as a risk averter, not as risk neutral. In Part I of this paper, however, we have shown that this conjecture is not a logical necessity. To assume that the consumer's preference ordering is expressed by the function G in the "First Step optimization", and then to assume that this consumer is risk neutral in the "Second Step", is completely compatible. By the nature of our "First Step optimization", the risk preference information, which might have been included in the function G, does not affect the result of the First Step.

We now examine if the consumption demand of such a consumer follows a martingale under "Stochastic Perfect Foresight".

Assuming (II : 10a), and making use of CEM, we first note that if the consumer is risk neutral, the consumption demand is :

$$C_T = k(r)[\mu(B_T)] \cdot \cdots\cdots\cdots\cdots\cdots\cdots \quad \text{(II : 10b)}^{31}.$$

This consumption function is composed by two linear transforms from B_T to C_T, i.e.,

$$B_T \Rightarrow \mu(B_T) \Rightarrow k(r)\mu(B_T).$$

Given the consumption function (II : 10b), we know from (II : 9a) that B_T dynamically behaves in the Stochastic Perfect Foresight, satisfying :

$$\begin{aligned} B_{T+1} - B_T &= rB_T + [\omega_{T+1} - W_T^{T+1}] - (1+r)k(r)\mu(B_T) \\ &= rB_T + [\omega_{T+1} - W_T^{T+1}] - (1+r)C_T \cdot \cdots\cdots\cdots \quad \text{(II : 11).} \end{aligned}$$

Notice that (II : 11) is meaningful only in period T+1 after the market wage ω has actually occurred as ω_{T+1}. Otherwise the stochastic variable B_{T+1} does not exist, because B_{T+1} (which by definition includes H_{T+1}) is a distribution composed by using ω_{T+1} as (one of the) information available then. We now operate μ over (II : 11) as the operator to take the mean, as of T+1, of the stochastic variables. Because ω_{T+1} is an observed value as of T+1, $\mu(\omega_{T+1}) = \omega_{T+1}$, and because $\mu(C_T) = C_T$ (i.e., C_T as determined by (II : 10b) is a deterministic value), this operation results in :

$$\mu(B_{T+1}) - \mu(B_T) = r\mu(B_T) + [\omega_{T+1} - \mu(W_T^{T+1})] - (1+r)C_T \cdot \cdots\cdots \quad \text{(II : 12).}$$

Substituting (II : 10b) into $\mu(B_{T+1})$ and $\mu(B_T)$, and rearranging the terms, we obtain :

$$C_{T+1} - C_T = (r - (1+r)k(r))C_T + k(r)[\omega_{T+1} - \mu(W_T^{T+1})] \cdot \cdots\cdots \quad \text{(II : 13)}$$

in "Stochastic Perfect Foresight".

Defining $\theta_T \equiv \mu_\omega - \mu(W_T^{T+1})$, and assuming that the market wage ω is stochastically distributed with $\omega = \mu_\omega + \varepsilon$, $\mu(\varepsilon) = 0$, generating successively the observed time series data $\{\cdot\cdot\omega_T, \omega_{T+1}, \cdot\cdot\}$ along the "vertical" time, (II : 13) is equivalently written as :

$$C_{T+1} - C_T = (r - (1+r)\,k(r))C_T + k(r)\,\theta_T + k(r)\varepsilon_{T+1} \cdot \cdots\cdots\cdots \quad \text{(II : 14)}^{32},$$

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31 k(r) is defined as $k(r) \equiv \frac{1+r-m}{1+r}$, where $m \equiv \left(\frac{1+r}{1+\delta}\right)^{\frac{1}{(1-\alpha)}}$, and notice that $C_T^{T+J} = m^J C_T$ $(J \geq 1)$. Further, in order for the consumption function (II : 10b) to be meaningful, both the discounted utility sum and the present value of the consumption must converge. For convergence, it must be true that $0 < (1+r)^\alpha < 1+\delta$, $(0 < \alpha < 1)$, a condition which we now assume satisfied. This condition implies $0 < m < 1+r$ *and* $0 < k(r) < 1$.

32 When one estimates the equation (II : 14), there is no strict reason to assume that $\theta_T \equiv \mu_\omega - \mu(W_T^{T+1}) = 0$ (i.e., $\mu_\omega = \mu(W_T^{T+1})$). The assumptions we have imposed so far are not sufficient to assume that $\theta_T \equiv \mu_\omega - \mu(W_T^{T+1}) = 0$. If

The equation (II : 14) applies to the risk neutral consumer with the homothetic preference ordering, behaving in the "Stochastic Perfect Forecast.". It shows that the consumption demand C_{T+1}, apart from the predetermined trend term (r-(1+r)k(r)) C_T+k(r) θ_T, is equal to the sum of C_T and k(r) ε_{T+1}, the latter term having been generated as the time series random term with mean zero. We will then be able to conclude that the consumption demand of this case shows the martingale property, apart from trend.

Notice the reason why the consumption time series shows the martingale property. One of the crucial reasons is the fact that (II : 12) includes ω_{T+1} as the contemporaneous external stochastic force to change the mean of B from $\mu(B_T)$ to $\mu(B_{T+1})$. Another crucial reason is that C (the deterministic consumption derived by CEM) becomes the linear mapping from the mean of the distribution of budget (i.e., μ(B)) because the consumer is risk neutral with homothetic preference ordering. In this case, the effect of ω_{T+1}, included both in (II : 11) and (II : 12), is proportionally transmitted to C, the consumption.

II-4 : (Summary and Conclusion)

The equation (II : 14) is a statistically refutable hypothesis based on which it is possible to judge whether the stochastic consumption process $\{C_T\}$, apart from trend, is a martingale. The sum of the first two terms of the RHS of (II : 14) is the trend of $\{C_T\}$ under "Stochastic Perfect Foresight". If this specification statistically explains the time series consumption data reasonably well, we may conclude that $\{C_T\}$ is in fact a martingale.

Notice, however, that the hypothesis (II : 14) is composed by assumptions including (a) a risk neutral consumer, (b) "Stochastic Perfect Foresight", (c) ω (i.e., "nature") is

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one assumes additionally that the market wage income is distributed i.i.d., and that the consumer uses the sample mean of the past observations of ω as the mean of the stochastic variable W_T^{T+1}, then the strong law of large numbers will suggest that it would be possible to assume μ_ω-$\mu(W_T^{T+1})=0$ in the long run. Under this strengthened assumption, (II : 14) is a theoretically refutable hypothesis to test whether the consumption obeys a random walk apart from the trend (r-(1+r) k(r)) C_T. It is in this case that we may assume the coincidence of ω, the "nature", and the subjective distribution W_T^{T+1}, as far as the first order moment is concerned, and we may call such a state as "rational expectation (RE)".

Because $\mu(W_T^{T+1})$ is a value composed by each consumer using all the information available by T, we could approximate θ_T by a function of all such information including ω_T, ω_{T-1}, ω_{T-2}, … etc. Not only the wage income but also other relevant variables could be tested as explanatory variables of θ_T. In estimating (II : 14) to examine the martingale property of consumption, there is a great degree of freedom for possible choice of the explanatory variables.

distributed as stochastic with the constant population mean μ_{ω}, and (d) the homothetic preference ordering implied by (II : 10a). If these assumptions are satisfied, the consumption time series will theoretically show the martingale property as (II : 14) implies.

What can one say generally about the property of the consumption stochastic process? We have shown above that the risk neutrality, among others, is one of the sufficient conditions for the time series consumption data to exhibit the martingale (or random walk) property. Even with the homothetic preference ordering (II : 10a), the consumption function of a non-risk-neutral consumer is :

$$C_T = k(r)[\widehat{B_T}],\ \widehat{B_T} \equiv Z^{-1}\ \mu Z[B_T],$$

where the transform $Z^{-1}\mu\ Z$ is non-linear. The dynamics of C in this case is not so simple as (II : 14). Even if B_T is a martingale in the Stochastic Perfect Foresight, the consumption path depends non-linearly on B_T, making the analysis of the time series consumption path apparently intractable. We are not certain if the consumption time series may show the martingale property in the risk non-neutrality case.

Apart from the risk neutrality, what is also essential is that the economy is under the Stochastic Perfect Foresight, and that the utility function is homothetic. Under these conditions the time series consumption process will show the martingale property apart from trend.

What we have discussed by this paper has to do with the sufficient conditions, not the necessary ones, under which the consumption time series may become martingale; we are well aware that this fact limits the scope of this paper. However, it is our understanding that many empirical studies about the subject of this paper have mainly dealt with possibilities such as the liquidity constraint, credit limits and so forth to explain the reason why the data do not apparently support the random walk-martingale hypothesis. This paper suggests, however, that the specification of the consumption function, including the risk preference, must also be thoroughly reexamined along with the possible market anomalies such as the liquidity constraint.

There are further implications of the analysis developed by this paper. In particular, our analysis has only dealt with the case where the interest rate variable (r) is deterministic. It is desirable then to extend the analysis to include the case in which r and $\overrightarrow{W_T}$ are jointly distributed. We have dealt with this issue in a separate paper (see

Mutoh–Tanaka[2015], Section 5). It is suggested there that the certainty equivalence concept is useful in resolving the missing equation difficulty of macroeconomic models.

(Tanaka, Former Specially Appointed Professor, Faculty of Economics, Seikei University
Mutoh, Professor, Faculty of Economics, Seikei University)

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